n-dimensional geometric-shifted global bilinear correspondences of Langlands on mixed motives — III

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Abstract

This third paper, devoted to global correspondences of Langlands, bears more particularly on geometric-shifted bilinear correspondences on mixed (bi)motives generated under the action of the products, right by left, of differential elliptic operators.

The mathematical frame, underlying these correspondences, deals with the categories of the Suslin-Voevodsky mixed (bi)motives and of the Chow mixed (bi)motives which are both in one-to-one correspondence with the functional representation spaces of the shifted algebraic bilinear semigroups.

A bilinear holomorphic and supercuspidal spectral representation of an elliptic bioperator is then developed.

Contents

Introduction

This paper constitutes the third part of the n-dimensional global correspondences of Langlands [Pie3], [Pie4] and is particularly devoted to the study of the Langlands correspondences on mixed (bi)motives.

When the n-dimensional global correspondences of Langlands [Cara] bear on pure bimotives, the related geometric-shifted correspondences deal with mixed bimotives which are assumed to be generated under the action of the products, right by left, of differential (elliptic) operators.

This leads us, more particularly, to:

a) work in the frame of the **categories of the Suslin-Voevodsky (mixed) (bi)motives and of the Chow (mixed) (bi)motives** which are both in one-to-one correspondence with the functional representation spaces of the (shifted) algebraic bilinear semigroups, as introduced in [Pie3].

b) envisage a **bilinear version for the index theorem**.

c) develop a **bilinear holomorphic and (super)cuspidal spectral representation of an elliptic bioperator**.

A first step then consists in introducing **triangulated categories of mixed (bi)motives** which are built on corresponding pure (bi)motives being in one-to-one correspondence with the **functional representation spaces of algebraic bilinear semigroups** recalled in chapter 1 and hereafter.

- Let F_ω (resp. $F_{\overline{\omega}}$) denote the set of left (resp. right) **pseudo-ramified complex completions** $F_{\omega_{j,m_j}}$ (resp. $F_{\overline{\omega}_{j,m_j}}$) corresponding to transcendental extensions restricted to the upper (resp. lower) half space and being in one-to-one correspondence with the corresponding complex splitting subsemifields $\widetilde{F}_{\omega_{j,m_j}}$ (resp. $\widetilde{F}_{\overline{\omega}_{j,m_j}}$) characterized by Galois extension degrees given by integers modulo N.

  Similarly, let F_v^+ (resp. $F_{\overline{v}}^+$) be the set of left (resp. right) **real pseudo-ramified completions** $F^+_{v_{j_\delta,m_{j_\delta}}}$ (resp. $F^+_{\overline{v}_{j_\delta,m_{j_\delta}}}$) in one-to-one correspondence with the corresponding real splitting subsemifields $\widetilde{F}^+_{v_{j_\delta,m_{j_\delta}}}$ (resp. $\widetilde{F}^+_{\overline{v}_{j_\delta,m_{j_\delta}}}$) of the (finite) extension semifield $\widetilde{F}_L^+$ (resp. $\widetilde{F}_R^+$) of a number field k of characteristic zero.

  The set F_v^+ (resp. $F_{\overline{v}}^+$) of left (resp. right) real pseudoramified completions covers the corresponding set F_ω (resp. $F_{\overline{\omega}}$) of complex completions.

- **The bilinear algebraic semigroup** of matrices $\mathrm{GL}_n(\widetilde{F}_{\overline{\omega}} \times \widetilde{F}_\omega) \equiv T_n^t(\widetilde{F}_{\overline{\omega}}) \times T_n(\widetilde{F}_\omega)$ has its representation space in the bilinear affine semigroup $G^{(2n)}(\widetilde{F}_{\overline{\omega}} \times \widetilde{F}_\omega)$ given by

the (bilinear) tensor product $\widetilde{M}_R^{2n}\otimes\widetilde{M}_L^{2n}$ of a right $T_n^t(\widetilde{F}_{\overline{\omega}})$-semimodule $\widetilde{M}_R^{2n}$ by its left equivalent $\widetilde{M}_L^{2n}$.

The bilinear algebraic semigroup $G^{(2n)}(\widetilde{F}_{\overline{\omega}}\times\widetilde{F}_{\omega})$, with entries in the product $\widetilde{F}_{\overline{\omega}}\times\widetilde{F}_{\omega}$ of right pseudoramified complex extensions $\widetilde{F}_{\overline{\omega}}$ by the left equivalent set $\widetilde{F}_{\omega}$, gives rise by an isomorphism of compactification to the complete algebraic bilinear semigroup $G^{(2n)}(F_{\overline{\omega}}\times F_{\omega})$ (which is an abstract bisemivariety) over the product $F_{\overline{\omega}}\times F_{\omega}$ of sets of completions.

The linear algebraic semigroup $G^{(2n)}(\widetilde{F}_{\omega})\equiv\widetilde{M}_L^{2n}$ (resp. $G^{(2n)}(\widetilde{F}_{\overline{\omega}})\equiv\widetilde{M}_R^{2n}$) decomposes into the set $\{\widetilde{g}_L^{(2n)}(j,m_j)\}_{j,m_j}$ (resp. $\{\widetilde{g}_R^{(2n)}(j,m_j)\}_{j,m_j}$) of r packets, $1\le j\le r$, of complex equivalent conjugacy class representatives $\widetilde{g}_L^{(2n)}(j,m_j)$ (resp. $\widetilde{g}_R^{(2n)}(j,m_j)$).

- The **functional representation space $\mathbf{FRepsp}(\mathbf{GL}_{\boldsymbol{n}}(\boldsymbol{F_{\overline{\omega}}}\times\boldsymbol{F_{\omega}}))$ of the complete bilinear semigroup $\mathbf{GL}_{\boldsymbol{n}}(\boldsymbol{F_{\overline{\omega}}}\times\boldsymbol{F_{\omega}})$** is the bisemisheaf $(\widehat{M}_R^{2n}\otimes\widehat{M}_L^{2n})$ of C^∞-differentiable bifunctions on $(M_R^{2n}\otimes M_L^{2n})$, i.e. the (tensor) product of the semisheaf $\widehat{M}_L^{2n}$ of C^∞-differentiable functions on M_L^{2n} by the semisheaf $\widehat{M}_R^{2n}$ of C^∞-differentiable cofunctions on M_R^{2n}.

- Let $\mathrm{CY}^{2n_\ell}(Y_L)\subset\cancel{Z}^{2n_\ell}(Y_L)\subset\mathrm{CH}^{2n_\ell}(Y_L)$ (resp. $\mathrm{CY}^{2n_\ell}(Y_R)\subset\cancel{Z}^{2n_\ell}(Y_R)\subset\mathrm{CH}^{2n_\ell}(Y_R)$) be a left (resp. right) algebraic semicycle of dimension $2n_\ell$ on the left (resp. right) algebraic semigroup $Y_L\equiv G^{(2n)}(\widetilde{F}_{\omega})$ (resp. $Y_R\equiv G^{(2n)}(\widetilde{F}_{\overline{\omega}})$) of complex dimension n, $n_\ell<n$, where $\cancel{Z}^{2n_\ell}(Y_L)$ (resp. $\cancel{Z}^{2n_\ell}(Y_R)$) is the semigroup of algebraic semicycles of codimension $2n_\ell$ and $\mathrm{CH}^{2n_\ell}(Y_L)$ (resp. $\mathrm{CH}^{2n_\ell}(Y_R)$) is the Chow semigroup of algebraic semicycles of codimension $2n_\ell$ on Y_L (resp. Y_R) [Jan1].

- In this context, it is recalled that **a Suslin-Voevodsky left (resp. right) presheaf $\boldsymbol{M(X_L^{\mathrm{sv}})}$ (resp. $\boldsymbol{M(X_R^{\mathrm{sv}})}$)** on the smooth semischeme X_L^{sv} (resp. X_R^{sv}) of complex dimension ℓ on the category $\mathrm{Sm}_L(k)$ (resp. $\mathrm{Sm}_R(k)$) of smooth semischemes over k is a functor from X_L^{sv} (resp. X_R^{sv}) to the chain complex associated with the abelian semigroup $\coprod\limits_{i_\ell}\mathrm{Hom}_{\mathrm{Sm}_L(k)}(\dot{\Sigma}_L,\mathrm{SP}^{i_\ell}(X_L^{\mathrm{sv}}))$ (resp. $\coprod\limits_{i_\ell}\mathrm{Hom}_{\mathrm{Sm}_R(k)}(\dot{\Sigma}_R,\mathrm{SP}^{i_\ell}(X_R^{\mathrm{sv}}))$) [Mor] where:
  - $\dot{\Sigma}_L$ (resp. $\dot{\Sigma}_R$) is a cosimplicial object from the collection of the left (resp. right) complex topological $2n_\ell$-simplices $\Sigma_L^{2n_\ell}$ (resp. $\Sigma_R^{2n_\ell}$).
  - $\mathrm{SP}^{i_\ell}(X_L^{\mathrm{sv}})$ denotes the i_ℓ-th symmetric product of X_L^{sv}.

A **Suslin-Voevodsky submotive** of dimension $2n_\ell = i_\ell \times 2\ell$ is noted $Z_L(2n_\ell)$ (resp. $Z_R(2n_\ell)$) and corresponds to a left (resp. right) semicycle $\mathrm{CY}^{2n_\ell}(Y_L)$ (resp. $\mathrm{CY}^{2n_\ell}(Y_R)$) in $\not Z^{2n_\ell}(Y_L)$ (resp. $\not Z^{2n_\ell}(Y_R)$).
In order to define Suslin-Voevodsky mixed motives, **shifted correspondences** must be introduced by the homomorphism:

$$\begin{array}{rll} \mathrm{CORR}^S_L : & \mathrm{Corr}(\mathrm{SP}^{f_\ell}(X^{\mathrm{sv}}_L), X_L^{2n_\ell - 2f_\ell\cdot\ell}) \longrightarrow & \mathrm{Corr}^S(\Delta_L^{2f_\ell\cdot\ell}, X_L^{2n_\ell-2f_\ell\cdot\ell}) \\ (\text{resp.} \quad \mathrm{CORR}^S_R : & \mathrm{Corr}(\mathrm{SP}^{f_\ell}(X^{\mathrm{sv}}_R), X_R^{2n_\ell - 2f_\ell\cdot\ell}) \longrightarrow & \mathrm{Corr}^S(\Delta_R^{2f_\ell\cdot\ell}, X_R^{2n_\ell-2f_\ell\cdot\ell})\ ) \end{array}$$

- from correspondences $\mathrm{Corr}(\mathrm{SP}^{f_\ell}(X^{\mathrm{sv}}_L), X_L^{2n_\ell - 2f_\ell\cdot\ell})$ sending the i_ℓ-th submotive $\mathrm{SP}^{i_\ell}(X^{\mathrm{sv}}_L)$ of dimension $2n_\ell = i_\ell \times 2\ell$ to the product $X_L^{2n_\ell} = \mathrm{SP}^{f_\ell}(X^{\mathrm{sv}}_L) \times X_L^{2n_\ell - 2f_\ell\cdot\ell}$ where $X_L^{2n_\ell - 2f_\ell\cdot\ell}$ is a smooth scheme of complex dimension $(n_\ell - f_\ell\cdot\ell)$, $f_\ell\cdot\ell \leq n_\ell \leq n$, $i_\ell \cdot \ell = n_\ell = (f_\ell \cdot \ell) + (n_\ell - f_\ell \cdot \ell)$,
- to shifted correspondences $\mathrm{Corr}^S(\Delta_L^{2f_\ell\cdot\ell}, X_L^{2n_\ell-2f_\ell\cdot\ell})$ where the smooth semischemes $\mathrm{SP}^{f_\ell}(X^{\mathrm{sv}}_L)$ has been sent to the corresponding smooth semischeme

$$\Delta_L^{2f_\ell\cdot\ell} = \mathrm{SP}^{f_\ell}(X^{\mathrm{sv}}_L) \times \mathrm{AdFRepsp}(T_{f_\ell\cdot\ell}(\mathbb{C}\,))$$

  where $\Delta_L^{2f_\ell\cdot\ell}$ is the total space of the tangent bundle $\mathrm{TAN}(\mathrm{SP}^{f_\ell}(X^{\mathrm{sv}}_L))$ with base space $\mathrm{SP}^{f_\ell}(X^{\mathrm{sv}}_L)$ and fibre given by the adjoint functional representation space of the group $T_{f_\ell\cdot\ell}(\mathbb{C}\,)$ of triangular matrices.

A **Suslin-Voevodsky left (resp. right) mixed semimotive** $M_{\mathrm{DM}_L(k)}(X^{\mathrm{sv}}_L)$ (resp. $M_{\mathrm{DM}_R(k)}(X^{\mathrm{sv}}_R)$) can be defined as the functor

$$\begin{array}{rl} M_{\mathrm{DM}_L(k)}(X^{\mathrm{sv}}_L) = & \displaystyle\coprod_{i_\ell, f_\ell} \mathrm{Hom}_{\mathrm{Sm}_L(k)}(\mathrm{SP}^{i_\ell}(X^{\mathrm{sv}}_L), X_L^{2n_\ell}[2f_\ell\cdot\ell]) \\ (\text{resp.} \quad M_{\mathrm{DM}_R(k)}(X^{\mathrm{sv}}_R) = & \displaystyle\coprod_{i_\ell, f_\ell} \mathrm{Hom}_{\mathrm{Sm}_R(k)}(\mathrm{SP}^{i_\ell}(X^{\mathrm{sv}}_R), X_R^{2n_\ell}[2f_\ell\cdot\ell])\ ) \end{array}$$

where:

- $X_L^{2n_\ell}[2f_\ell\cdot\ell] = \Delta_L^{2f_\ell\cdot\ell} \times X_L^{2n_\ell-2f_\ell\cdot\ell}$ is the smooth semischeme of dimension $2n_\ell$ shifted in $2f_\ell\cdot\ell$ dimensions by means of the shifted correspondences sending $\mathrm{SP}^{f_\ell}(X^{\mathrm{sv}}_L)$ into $\Delta_L^{2f_\ell\cdot\ell}$.
- $\mathrm{DM}_L(k)$ is the triangulated category of Suslin-Voevodsky left mixed semimotives.

The following **proposition** can then be stated (**propositions 2.11 and 2.12**):

Under the action of the adjoint functional representation space

$$\mathrm{AdFRepsp}(\mathrm{GL}_{f_\ell\cdot\ell}(\mathbb{C}\times\mathbb{C})) = \mathrm{AdFRepsp}(T_{f_\ell\cdot\ell}(\mathbb{C})) \times \mathrm{AdFRepsp}(T^t_{f_\ell\cdot\ell}(\mathbb{C}))\ ,$$

the bilinear cohomology of the Suslin-Voevodsky pure bimotive $M(X^{\mathrm{sv}}_{R\times L}) = M(X^{\mathrm{sv}}_R)\otimes M(X^{\mathrm{sv}}_L)$ is transformed into the **the bilinear cohomology of the corresponding Suslin-Voevodsky mixed bimotive** $M_{\mathrm{DM}_{R\times L}}(X^{\mathrm{sv}}_{R\times L}) = M_{\mathrm{DM}_R}(X^{\mathrm{sv}}_R)\otimes M_{\mathrm{DM}_L}(X^{\mathrm{sv}}_L)$ by the isomorphism:

$$\begin{aligned}\mathbb{H}\,D_{2f_\ell\cdot\ell}: \quad & H^{2n_\ell}(M(X^{\mathrm{sv}}_{R\times L}), Z_{R\times L}(2n_\ell))\\ & \qquad\longrightarrow H^{2n_\ell-2f_\ell\cdot\ell}(M_{\mathrm{DM}_{R\times L}}(X^{\mathrm{sv}}_{R\times L}), Z_{R\times L}(2n_\ell[2f_\ell\cdot\ell]))\end{aligned}$$

where:

- $Z_{R\times L}(2n_\ell) = Z_R(2n_\ell)\times Z_L(2n_\ell)$ is the product of Suslin-Voevodsky pure semisubmotives of dimension $2n_\ell$;
- $Z_{R\times L}(2n_\ell[2f_\ell\cdot\ell]) = Z_R(2n_\ell[2f_\ell\cdot\ell])\times Z_L(2n_\ell[2f_\ell\cdot\ell]) \equiv X^{2n_\ell}_R[2f_\ell\cdot\ell])\times X^{2n_\ell}_L[2f_\ell\cdot\ell]$ is the product of Suslin-Voevodsky mixed subsemimotives of dimension $2n_\ell$ shifted in $2f_\ell\cdot\ell$ dimensions;
- $H^{2n_\ell-2f_\ell\cdot\ell}(\centerdot,\centerdot)$ is the mixed bilinear cohomology defined in proposition 2.17;

in such a way that:

a) $H^{2n_\ell-2f_\ell\cdot\ell}(M_{\mathrm{DM}_{R\times L}}(X^{\mathrm{sv}}_{R\times L}), Z_{R\times L}(2n_\ell[2f_\ell\cdot\ell])) = H_{2f_\ell\cdot\ell}(\Delta^{2f_\ell\cdot\ell}_R\times\Delta^{2f_\ell\cdot\ell}_L, \mathcal{F}^{2f_\ell\cdot\ell}_{R\times L}(\mathrm{TAN}))$ $\times H^{2n_\ell}(M(X^{\mathrm{sv}}_{R\times L}), Z_{R\times L}(2n_\ell))$ where $\mathcal{F}^{2f_\ell\cdot\ell}_{R\times L}(\mathrm{TAN})$ is the bilinear fibre of the tangent bibundle $\mathrm{TAN}(\mathrm{SP}^{f_\ell}(X^{\mathrm{sv}}_{R\times L})) = \mathrm{TAN}(\mathrm{SP}^{f_\ell}(X^{\mathrm{sv}}_R))\times\mathrm{TAN}(\mathrm{SP}^{f_\ell}(X^{\mathrm{sv}}_L))$;

b) $H^{2n_\ell-2f_\ell\cdot\ell}(M_{\mathrm{DM}_{\mathrm{R}\times\mathrm{L}}}(X^{\mathrm{sv}}_{R\times L}), Z_{R\times L}(2n_\ell[2f_\ell\cdot\ell])) \simeq \mathrm{FRepsp}(\mathrm{GL}_{n_\ell[f_\ell\cdot\ell]}((F_{\overline{\omega}}\otimes\mathbb{C})\times(F_\omega\otimes\mathbb{C})))$ where $\mathrm{FRepsp}(\mathrm{GL}_{n_\ell[f_\ell\cdot\ell]}((F_{\overline{\omega}}\otimes\mathbb{C})\times(F_\omega\otimes\mathbb{C}))) = \mathrm{AdFRepsp}(\mathrm{GL}_{f_\ell\cdot\ell}(\mathbb{C}\otimes\mathbb{C}))$ $\times\mathrm{FRepsp}(\mathrm{GL}_{n_\ell}(F_{\overline{\omega}}\times F_\omega))$ is the functional representation space of the bilinear complete semigroup $\mathrm{GL}_{n_\ell[f_\ell\cdot\ell]}((F_{\overline{\omega}}\otimes\mathbb{C})\times(F_\omega\otimes\mathbb{C}))$ shifted in $(2f_\ell\cdot\ell)$ complex dimensions.

To be more explicit, let $D^{2f_\ell\cdot\ell}_R\otimes D^{2f_\ell\cdot\ell}_L$ be the product of a (right) differential (elliptic) operator $D^{2f_\ell\cdot\ell}_R$ acting on $2f_\ell\cdot\ell$ variables by its left equivalent. **This bioperator is defined by its action**:

$$D^{2f_\ell\cdot\ell}_R\otimes D^{2f_\ell\cdot\ell}_L: \qquad Z_R(2n_\ell)\times Z_L(2n_\ell) \longrightarrow Z_R(2n_\ell[2f_\ell\cdot\ell])\times Z_L(2n_\ell[2f_\ell\cdot\ell])$$

transforming the Suslin-Voevodsky pure subbisemimotive of dimension $(2n_\ell)$ into the corresponding Suslin-Voevodsky mixed subbisemimotive of dimension $(2n_\ell)$ shifted in $(2f_\ell\cdot\ell)$ dimensions.

Indeed, it is seen in chapter 2 that $H_{2f_\ell\cdot\ell}(\Delta_R^{2f_\ell\cdot\ell}\times\Delta_L^{2f_\ell\cdot\ell},\mathcal{F}_{R\times L}^{2f_\ell\cdot\ell}(\mathrm{TAN}))\simeq \mathrm{AdFRepsp}(\mathrm{GL}_{f_\ell\cdot\ell}(\mathbb{C}\times\mathbb{C}))$ is the bilinear homology with coefficients in the bilinear fibre $\mathcal{F}_{R\times L}^{2f_\ell\cdot\ell}(\mathrm{TAN})$ of the tangent bibundle $\mathrm{TAN}[\mathrm{SP}^{f_\ell}(X_R^{\mathrm{sv}})\times\mathrm{SP}^{f_\ell}(X_L^{\mathrm{sv}})]$.

In connection with the work of G. Kasparov [Kas], we shall introduce in chapter 3 **a $\boldsymbol{K_*K^*}$ functor** on the categories of elliptic bioperators and products, right by left, of Suslin-Voevodsky pure motives allowing to set up a **bilinear version of the index theorem**.

a) If $H^*(M(X^{\mathrm{sv}}_{R\times L}))=\bigoplus\limits_{n_\ell=n_1}^{n_s}H^{2n_\ell}(M(X^{\mathrm{sv}}_{R\times L}),Z_{R\times L}(2n_\ell))$ denotes the total bilinear cohomology of the pure bimotive $M(X^{\mathrm{sv}}_{R\times L})$ and if $K^*(X^{\mathrm{sv}}_{R\times L})$, introduced as the product, right by left, of abelian semigroups generated by the complex vector bundles over $X^{\mathrm{sv}}_{R\times L}=X^{\mathrm{sv}}_R\times X^{\mathrm{sv}}_L$, is the K-cohomology associated with the pure bimotive $M(X^{\mathrm{sv}}_{R\times L})$, **the total Chern character in the bilinear $\boldsymbol{K}$-cohomology [W-R] of the pure bimotive $\boldsymbol{M(X^{\mathrm{sv}}_{R\times L})}$** is given by the homomorphism:

$$\mathrm{ch}^*(M(X^{\mathrm{sv}}_{R\times L})):\qquad K^*(X^{\mathrm{sv}}_{R\times L})\longrightarrow H^*(M(X^{\mathrm{sv}}_{R\times L}))\ .$$

b) Similarly, if $H_*(\Delta_R^*\times\Delta_L^*,\mathcal{F}^*_{R\times L}(\mathrm{TAN}))=\bigoplus\limits_{f_\ell\cdot\ell}H_{2f_\ell\cdot\ell}(\Delta_R^{2f_\ell\cdot\ell}\times\Delta_L^{2f_\ell\cdot\ell},\mathcal{F}_{R\times L}^{2f_\ell\cdot\ell}(\mathrm{TAN}))\simeq \bigoplus\limits_{f_\ell\cdot\ell}\mathrm{AdFRepsp}(\mathrm{GL}_{f_\ell\cdot\ell}(\mathbb{C}\times\mathbb{C}))$ is the total bilinear homology with coefficients in the set of bilinear fibres $\mathcal{F}_{R\times L}^{2f_\ell\cdot\ell}(\mathrm{TAN})$ and if $K_*(\mathrm{SP}^{\mathrm{FL}}(X^{\mathrm{sv}}_{R\times L}))$ is the bilinear K-homology, introduced as the product, right by left, of abelian semigroups generated by the set of tangent bibundles $\mathrm{TAN}(\mathrm{SP}^{f_\ell}(X^{\mathrm{sv}}_{R\times L}))$, **the Chern character in the bilinear $\boldsymbol{K}$-homology, associated with the pure bimotive $\boldsymbol{M(X^{\mathrm{sv}}_{R\times L})}$**, is given by the homomorphism:

$$\mathrm{ch}_*(M(X^{\mathrm{sv}}_{R\times L})):\qquad K_*(\mathrm{SP}^{\mathrm{FL}}(X^{\mathrm{sv}}_{R\times L}))\longrightarrow H_*(\Delta_R^*\times\Delta_L^*,\mathcal{F}^*_{R\times L}(\mathrm{TAN}));$$

c) **The total Chern character $\boldsymbol{\mathrm{ch}^*(M_{\mathrm{DM}_{R\times L}}(X^{\mathrm{sv}}_{R\times L}))}$ of the Suslin-Voevodsky mixed bisemimotive $\boldsymbol{M_{\mathrm{DM}_{R\times L}}(X^{\mathrm{sv}}_{R\times L})}$** in the mixed bilinear K-homology-K-cohomology is given by the homomorphism:

$$\begin{aligned}\mathrm{ch}^*(M_{\mathrm{DM}_{R\times L}}(X^{\mathrm{sv}}_{R\times L})):\quad & K_*(\mathrm{SP}^{\mathrm{FL}}(X^{\mathrm{sv}}_{R\times L}))\times K^*(X^{\mathrm{sv}}_{R\times L})\\ &\longrightarrow H_*(\Delta_R^*\times\Delta_L^*,\mathcal{F}^*_{R\times L}(\mathrm{TAN}))\times H^*(M(X^{\mathrm{sv}}_{R\times L}));\end{aligned}$$

in such a way that

$$\mathrm{ch}_*(M(X^{\mathrm{sv}}_{R\times L})\times\mathrm{ch}^*(M(X^{\mathrm{sv}}_{R\times L})\longrightarrow\mathrm{ch}^*(M_{\mathrm{DM}_{R\times L}}(X^{\mathrm{sv}}_{R\times L}))$$

corresponds to a bilinear version of the index theorem.

Chapter 4 deals with the **holomorphic and toroidal spectral representations of an elliptic bioperator** associated with the functional representation space $\mathrm{FRepsp}(\mathrm{GL}_{n_\ell[f_\ell\cdot\ell]}(F_{\overline{\omega}}\otimes\mathbb{C}\,)\times(F_\omega\otimes\mathbb{C}\,))$ of the complete bilinear semigroup $\mathrm{GL}_{n_\ell[f_\ell\cdot\ell]}(F_{\overline{\omega}}\otimes\mathbb{C}\,)\times(F_\omega\otimes\mathbb{C}\,)$ shifted in $(2f_\ell\cdot\ell)$ dimensions.
Taking into account that:

1) the functional representation space [Del1], [Vog] $\mathrm{FRepsp}(\mathrm{GL}_{n_\ell}(F_{\overline{\omega}}\otimes F_\omega))$ of the complete bilinear semigroup $\mathrm{GL}_{n_\ell}(F_{\overline{\omega}}\times F_\omega)$ is the bisemisheaf $(\widehat{M}^{2n_\ell}_R\otimes\widehat{M}^{2n_\ell}_L)$ of differentiable bifunctions over $\mathrm{GL}_{n_\ell}(F_{\overline{\omega}}\times(F_\omega)$,

2) there exists a toroidal isomorphism of compactification:

$$\gamma_{R\times L}:\qquad \widehat{M}^{2n_\ell}_R\otimes\widehat{M}^{2n_\ell}_L\longrightarrow \widehat{M}^{2n_\ell}_{T_R}\otimes\widehat{M}^{2n_\ell}_{T_L}$$

   sending $(\widehat{M}^{2n_\ell}_R\otimes\widehat{M}^{2n_\ell}_L)$ into its toroidal equivalent $(\widehat{M}^{2n_\ell}_{T_R}\otimes\widehat{M}^{2n_\ell}_{T_L})=\mathrm{FRepsp}(\mathrm{GL}_{n_\ell}(F^T_{\overline{\omega}}\times F^T_\omega))$ where F^T_ω and $F^T_{\overline{\omega}}$ are sets of toroidal completions,

3) there exists a correspondence:

$$\mathbb{H}_\oplus:\qquad \widehat{M}^{2n_\ell}_{T_R}\otimes\widehat{M}^{2n_\ell}_{T_L}\longrightarrow \widehat{M}^{2n_\ell}_{T_{R_\oplus}}\otimes\widehat{M}^{2n_\ell}_{T_{L_\oplus}}$$

   in such a way that

$$\widehat{M}^{2n_\ell}_{T_{R_\oplus}}\otimes\widehat{M}^{2n_\ell}_{T_{L_\oplus}}=\bigoplus_j\bigoplus_{m_j}(\widehat{M}^{2n_\ell}_{T_{\overline{\omega}_{j,m_j}}}\otimes\widehat{M}^{2n_\ell}_{T_{\omega_{j,m_j}}})$$

   decomposes into the sum of bisections $(\widehat{M}^{2n_\ell}_{T_{\overline{\omega}_{j,m_j}}}\otimes\widehat{M}^{2n_\ell}_{T_{\omega_{j,m_j}}})$ of $(\widehat{M}^{2n_\ell}_{T_R}\otimes\widehat{M}^{2n_\ell}_{T_L})$ according to the conjugacy class representatives of $(\widehat{M}^{2n_\ell}_{T_R}\otimes\widehat{M}^{2n_\ell}_{T_L})$,

the elliptic bioperator $(D^{2f_\ell\cdot\ell}_R\otimes D^{2f_\ell\cdot\ell}_L)$ maps $(\widehat{M}^{2n_\ell}_{T_{R_\oplus}}\otimes\widehat{M}^{2n_\ell}_{T_{L_\oplus}})$ into its shifted equivalent according to:

$$D^{2f_\ell\cdot\ell}_R\otimes D^{2f_\ell\cdot\ell}_L:\qquad \widehat{M}^{2n_\ell}_{T_{R_\oplus}}\otimes\widehat{M}^{2n_\ell}_{T_{L_\oplus}}\longrightarrow \widehat{M}^{2n_\ell}_{T_{R_\oplus}}[2f_\ell\cdot\ell]\otimes\widehat{M}^{2n_\ell}_{T_{L_\oplus}}[2f_\ell\cdot\ell]$$

where:

$$(\widehat{M}^{2n_\ell}_{T_R}[2f_\ell\cdot\ell]\otimes\widehat{M}^{2n_\ell}_{T_L}[2f_\ell\cdot\ell])=\mathrm{FRepsp}(\mathrm{GL}_{n_\ell[f_\ell\cdot\ell]}((F^T_{\overline{\omega}}\otimes\mathbb{C}\,)\times((F^T_\omega\otimes\mathbb{C}\,))$$

is the perverse bisemisheaf of differentiable bifunctions over $\mathrm{GL}_{n_\ell[f_\ell\cdot\ell]}((F^T_{\overline{\omega}}\otimes\mathbb{C}\,)\times((F^T_\omega\otimes\mathbb{C}\,))$ shifted in $(2f_\ell\cdot\ell)$ dimensions: it is thus a $(\mathcal{D}_R\otimes\mathcal{D}_L)$-bisemimodule in such a way that $\mathcal{D}_R$ (resp. $\mathcal{D}_L$) is a right (resp. left) sheaf of differentiable operators of finite order with holomorphic coefficients [M-T].

- Referring to [Pie3], we see that each bifunction $(\widehat M^{2n_\ell}_{T_{\overline\omega_{j,m_j}}}\otimes \widehat M^{2n_\ell}_{T_{\overline\omega_{j,m_j}}})\in(\widehat M^{2n_\ell}_{T_{R_\oplus}}\otimes \widehat M^{2n_\ell}_{T_{L_\oplus}})$ is the product, right by left , of n_ℓ-dimensional complex semitori:

$$
\begin{aligned}
\widehat M^{2n_\ell}_{T_{\overline\omega_{j,m_j}}} &= T^{2n_\ell}_R(j,m_j)=\lambda^{\frac12}(2n_\ell,j,m_j)\ e^{-2\pi ijz_{n_\ell}}\\
\text{and}\quad \widehat M^{2n_\ell}_{T_{\omega_{j,m_j}}} &= T^{2n_\ell}_L(j,m_j)=\lambda^{\frac12}(2n_\ell,j,m_j)\ e^{2\pi ijz_{n_\ell}}\ ,\quad z_{n_\ell}\in\mathbb{C}^{\,n_\ell}\ ,
\end{aligned}
$$

in such a way that $\widehat M^{2n_\ell}_{T_{L_\oplus}}$ (resp. $\widehat M^{2n_\ell}_{T_{R_\oplus}}$) is the (truncated) **Fourier development of a normalized $2n_\ell$-dimensional left (resp. right) cusp form of weight 2** restricted to the upper (resp. lower) half space:

$$
\begin{aligned}
\widehat M^{2n_\ell}_{T_{L_\oplus}} &\equiv \mathrm{EIS}_L(2n_\ell,j,m_j)=\mathop{\oplus}_{j=1}^{r}\mathop{\oplus}_{m_j}\lambda^{\frac12}(2n_\ell,j,m_j)\ e^{2\pi ijz_{n_\ell}}\\
(\text{resp.}\quad \widehat M^{2n_\ell}_{T_{R_\oplus}} &\equiv \mathrm{EIS}_R(2n_\ell,j,m_j)=\mathop{\oplus}_{j=1}^{r}\mathop{\oplus}_{m_j}\lambda^{\frac12}(2n_\ell,j,m_j)\ e^{-2\pi ijz_{n_\ell}}\ ).
\end{aligned}
$$

- Similarly, $\widehat M^{2n_\ell}_{T_{L_\oplus}}[2f_\ell\cdot\ell]$ (resp. $\widehat M^{2n_\ell}_{T_{R_\oplus}}[2f_\ell\cdot\ell]$) decomposes into sums of $2n_\ell$-dimensional semitori shifted in $2f_\ell\cdot\ell$ dimensions in such a way that:

$$
\begin{aligned}
\widehat M^{2n_\ell}_{T_{L_\oplus}}[2f_\ell\cdot\ell] &\equiv \mathrm{EIS}_L(2n_\ell[2f_\ell\cdot\ell],j,m_j)\\
&= \mathop{\oplus}_{j=1}^{r}\mathop{\oplus}_{m_j} E_{2f_\ell\cdot\ell}(2n_\ell,j,m_j)\,.\,\lambda^{\frac12}(2n_\ell,j,m_j)\ e^{2\pi ijz_{n_\ell}}\\
(\text{resp.}\quad \widehat M^{2n_\ell}_{T_{R_\oplus}}[2f_\ell\cdot\ell] &\equiv \mathrm{EIS}_R(2n_\ell[2f_\ell\cdot\ell],j,m_j)\\
&= \mathop{\oplus}_{j=1}^{r}\mathop{\oplus}_{m_j} E_{2f_\ell\cdot\ell}(2n_\ell,j,m_j)\,.\,\lambda^{\frac12}(2n_\ell,j,m_j)\ e^{-2\pi ijz_{n_\ell}}\ )
\end{aligned}
$$

be the (truncated) **Fourier development of a normalized left (resp. right) $2n_\ell$-dimensional mixed cusp form shifted in $2f_\ell\cdot\ell$ dimensions, where $E_{2f_\ell\cdot\ell}(2n_\ell,j,m_j)$ are shifts of generalized global Hecke characters $\lambda^{\frac12}(2n_\ell,j,m_j)$.**

- This allows to set up the **bieigenvalue equation**:

$$
\begin{aligned}
&(D^{2f_\ell\cdot\ell}_R\otimes D^{2f_\ell\cdot\ell}_L)(\mathrm{EIS}_R(2n_\ell,j^{\mathrm{up}}=j,m_j))\otimes(\mathrm{EIS}_L(2n_\ell,j^{\mathrm{up}}=j,m_j))\\
&\qquad = E^2_{2f_\ell\cdot\ell}(2n_\ell,j,m_j)(\mathrm{EIS}_R(2n_\ell,j^{\mathrm{up}}=j,m_j))\otimes(\mathrm{EIS}_L(2n_\ell,j^{\mathrm{up}}=j,m_j))
\end{aligned}
$$

of which spectral representation is given by the set of r-bituples:

$$
\begin{aligned}
\big\{&(\mathrm{EIS}_R(2n_\ell,j^{\mathrm{up}}=1,m_1))\otimes(\mathrm{EIS}_L(2n_\ell,j^{\mathrm{up}}=1,m_1)),\cdots,\\
&\quad(\mathrm{EIS}_R(2n_\ell,j^{\mathrm{up}}=j,m_j))\otimes(\mathrm{EIS}_L(2n_\ell,j^{\mathrm{up}}=j,m_j)),\cdots,\\
&\qquad(\mathrm{EIS}_R(2n_\ell,j^{\mathrm{up}}=r,m_r))\otimes(\mathrm{EIS}_L(2n_\ell,j^{\mathrm{up}}=r,m_r))\big\}
\end{aligned}
$$

where $\mathbf{EIS}_L(2n_\ell, j^{\mathrm{up}} = j, m_j))$ (resp. $\mathbf{EIS}_R(2n_\ell, j^{\mathrm{up}} = j, m_j)))$ is the truncated Fourier development at the j classes of the $2n_\ell$-dimensional cusp form.

- It then appears that $\mathrm{EIS}_R(2n_\ell[2f_\ell\cdot\ell], j, m_j)\otimes \mathrm{EIS}_L(2n_\ell[2f_\ell\cdot\ell], j, m_j)$ constitutes a **supercuspidal representation of the shifted algebraic complete semigroup $\mathbf{GL}_{n_\ell[f_\ell\cdot\ell]}((F_{\overline{\omega}_\oplus}\otimes\mathbb{C})\times(F_{\omega_\oplus}\otimes\mathbb{C}))$.**

The origin of the (bilinear) spectral theory then results from geometric-shifted global (bilinear) correspondences of Langlands as it will be seen hereafter.

This leads us to develop **in chapter 5 geometric-shifted global bilinear correspondences of Langlands**.

- If $(W^{ab}_{F^{S_{\mathbb{C}}}_{\overline{\omega}}}\times W^{ab}_{F^{S_{\mathbb{C}}}_{\omega}})$ is the product, right by left, of the shifted global Weil group $W^{ab}_{F^{S_{\mathbb{C}}}_{\overline{\omega}}}$ and $W^{ab}_{F^{S_{\mathbb{C}}}_{\omega}}$ introduced in chapter 1, there exists an irreducible representation $\mathrm{IrrRep}^{(2n_\ell[2f_\ell\cdot\ell])}_{W_{F_{R\times L}}}(W^{ab}_{F^{S_{\mathbb{C}}}_{\overline{\omega}}}\times W^{ab}_{F^{S_{\mathbb{C}}}_{\omega}})$ of $(W^{ab}_{F^{S_{\mathbb{C}}}_{\overline{\omega}}}\times W^{ab}_{F^{S_{\mathbb{C}}}_{\omega}})$ given by the representation space

$$\begin{aligned}\mathrm{Repsp}(\mathrm{GL}_{n_\ell[f_\ell\cdot\ell]}((F_{\overline{\omega}_\oplus}\otimes\mathbb{C})\times(F_{\omega_\oplus}\otimes\mathbb{C}))) &\equiv G^{(2n_\ell[2f_\ell\cdot\ell])}((F_{\overline{\omega}_\oplus}\otimes\mathbb{C})\times(F_{\omega_\oplus}\otimes\mathbb{C}))\\ &\equiv M^{2n_\ell}_{R_\oplus}[2f_\ell\cdot\ell]\otimes M^{2n_\ell}_{L_\oplus}[2f_\ell\cdot\ell]\end{aligned}$$

  of the shifted bilinear complete semigroup $\mathrm{GL}_{n_\ell[f_\ell\cdot\ell]}((F_{\overline{\omega}_\oplus}\otimes\mathbb{C})\times(F_{\omega_\oplus}\otimes\mathbb{C}))$.
  So, **on the shifted irreducible bilinear complete semigroup $G^{(2n_\ell[2f_\ell\cdot\ell])}((F_{\overline{\omega}_\oplus}\otimes\mathbb{C})\times(F_{\omega_\oplus}\otimes\mathbb{C}))$, the geometric-shifted global bilinear correspondence of Langlands is**:

$$\begin{tikzcd}
\mathrm{IrrRep}^{(2n_\ell[2f_\ell\cdot\ell])}_{W_{F_{R\times L}}}(W^{ab}_{F^{S_{\mathbb{C}}}_{\overline{\omega}}}\times W^{ab}_{F^{S_{\mathbb{C}}}_{\omega}}) \arrow[r, "\sim"] \arrow[d, equal] & \mathrm{Irrcusp}(\mathrm{GL}_{n_\ell[f_\ell\cdot\ell]}((F^T_{\overline{\omega}}\otimes\mathbb{C})\times(F^T_{\omega}\otimes\mathbb{C}))) \arrow[d, equal] \\
G^{(2n_\ell[2f_\ell\cdot\ell])}((F_{\overline{\omega}_\oplus}\otimes\mathbb{C})\times(F_{\omega_\oplus}\otimes\mathbb{C})) \arrow[dr, "\sim"] & \mathrm{EIS}_{R\times L}(2n_\ell[2f_\ell\cdot\ell], j, m_j) \\
& G^{(2n_\ell[2f_\ell\cdot\ell])}((F^T_{\overline{\omega}}\otimes\mathbb{C})\times(F^T_{\omega}\otimes\mathbb{C})) \arrow[u, "\wr"]
\end{tikzcd}$$

  where $\mathrm{Irrcusp}(\mathrm{GL}_{n_\ell[f_\ell\cdot\ell]}((F^T_{\overline{\omega}}\otimes\mathbb{C})\times(F^T_{\omega}\otimes\mathbb{C})))$ is the shifted irreducible supercuspidal representation of $\mathrm{GL}_{n_\ell[f_\ell\cdot\ell]}((F^T_{\overline{\omega}}\otimes\mathbb{C})\times(F^T_{\omega}\otimes\mathbb{C}))$ over the product of toroidal shifted completions.

- Similarly, **on the reducible shifted $2n$-dimensional bilinear complete algebraic semigroup** $G^{2n[2f_n\cdot\ell]}((F_{\overline{\omega}_\oplus}\otimes\mathbb{C})\times(F_{\omega_\oplus}\otimes\mathbb{C}))=\underset{n_\ell=n_1}{\overset{n_s}{\boxplus}} G^{2n_\ell[2f_\ell\cdot\ell]}((F_{\overline{\omega}_\oplus}\otimes\mathbb{C})\times(F_{\omega_\oplus}\otimes\mathbb{C}))$, there exists the geometric-shifted global bilinear reducible correspondence of Langlands:

$$\begin{tikzcd}
\mathrm{RedRep}^{(2n[2f_n\cdot\ell])}_{W_{F_{R\times L}}}(W^{ab}_{F^{S_{\mathbb{C}}}_{\overline{\omega}}}\times W^{ab}_{F^{S_{\mathbb{C}}}_{\omega}}) \arrow[rr, "\sim"] \arrow[d, equal] & & \mathrm{Redcusp}(\mathrm{GL}_{n[f_n\cdot\ell]}((F^T_{\overline{\omega}}\otimes\mathbb{C})\times(F^T_{\omega}\otimes\mathbb{C}))) \arrow[d, equal] \\
G^{(2n[2f_n\cdot\ell])}((F_{\overline{\omega}_\oplus}\otimes\mathbb{C})\times(F_{\omega_\oplus}\otimes\mathbb{C})) \arrow[dr, "\sim"] & & \mathrm{EIS}_{R\times L}(2n_\ell[2f_\ell\cdot\ell],j,m_j) \\
& G^{(2n[2f_n\cdot\ell])}((F^T_{\overline{\omega}}\otimes\mathbb{C})\times(F^T_{\omega}\otimes\mathbb{C})) \arrow[ur, "\wr"] &
\end{tikzcd}$$

- Geometric-shifted global bilinear correspondences of Langlands are also established on real shifted irreducible and reducible bilinear complete algebraic semigroups.

- Remark that the geometric-shifted global bilinear correspondences of Langlands considered in this paper differ from **the geometric correspondences** initiated by V. Drinfeld and G. Laumon.

  Indeed, these deal with an ℓ-adic n-dimensional irreducible local system E on a smooth algebraic curve X over a ground field K and say that it can be associated to E an automorphic sheaf S_E which is a perverse sheaf on the moduli stack $\mathrm{Bun}_n(X)$ of vector bundles of rank n on X [Lau], [F-G-V], [Fre], [Gai].

- The last version of this paper was motivated to precise the nature of a general bilinear mixed cohomology theory.

1 Global class field concepts and pure motivic cohomologies

1.1 Pseudo-ramified and pseudo-unramified infinite places of semifields

- Let k be a number field of characteristic 0 and let $\widetilde{F}$ denote a finite extension set of k such that $\widetilde{F}$ is assumed to be a **symmetric splitting field** $\widetilde{F}=\widetilde{F}_R\cup\widetilde{F}_L$ composed of the right and left algebraic extension semifields $\widetilde{F}_R$ and $\widetilde{F}_L$ in one-to-one correspondence.

- $\widetilde{F}_L$ (resp. $\widetilde{F}_R$) is assumed to be composed of a set of complex (resp. conjugate complex) simple roots of a polynomial ring over k. If the algebraic extension field of k is real, then the symmetric splitting field will be noted $\widetilde{F}^+=\widetilde{F}^+_R\cup\widetilde{F}^+_L$ where the left (resp. right) algebraic extension semifield $\widetilde{F}^+_L$ (resp. $\widetilde{F}^+_R$) is composed of the set of positive (resp. symmetric negative) simple real roots.

- The left and right equivalence classes of the global completions of $\widetilde{F}^{(+)}_L$ and $\widetilde{F}^{(+)}_R$ (which correspond to transcendental extensions of k), obtained by an isomorphism of compactification of the corresponding finite extensions, are the left and right infinite real (resp. complex) places of $F^{(+)}_L$ and $F^{(+)}_R$: they are noted $v=\{v_1,\dots,v_{j_\delta},\dots,v_{r_\delta}\}$ and $\overline{v}=\{\overline{v}_1,\dots,\overline{v}_{j_\delta},\dots,\overline{v}_{r_\delta}\}$ in the real case and $\omega=\{\omega_1,\dots,\omega_j,\dots,\omega_r\}$ and $\overline{\omega}=\{\overline{\omega}_1,\dots,\overline{\omega}_j,\dots,\overline{\omega}_r\}$ in the complex case.

- **The pseudo-unramified real places** are characterized algebraically by their global class residue degrees $f_{v_{j_\delta}}$ and $f_{\overline{v}_{j_\delta}}$ given by $f_{v_{j_\delta}}=[\widetilde{F}^{+,nr}_{v_{j_\delta}}:k]=j$ and $f_{\overline{v}_{j_\delta}}=[\widetilde{F}^{+,nr}_{\overline{v}_{j_\delta}}:k]=j$, $j\in I\!N$, $1\le j_\delta\le r_\delta$, where $\widetilde{F}^{+,nr}_{v_{j_\delta}}$ and $\widetilde{F}^{+,nr}_{\overline{v}_{j_\delta}}$ denote basic real pseudo-unramified extensions (splitting subsemifields) of k in one-to-one correspondence with the corresponding completions $F^{+,nr}_{v_{j_\delta}}$ and $F^{+,nr}_{\overline{v}_{j_\delta}}$ at the places v_{j_δ} and $\overline{v}j_\delta$.

  Similarly, **pseudo-unramified complex places** are characterized by their global class residue degrees f_{ω_j} and $f_{\overline{\omega}_j}$ given by:

$$f_{\omega_j}=[\widetilde{F}^{nr}_{\omega_j}:k]=j \qquad \text{and} \qquad f_{\overline{\omega}_j}=[\widetilde{F}^{nr}_{\overline{\omega}_j}:k]=j$$

  where $\widetilde{F}^{nr}_{\omega_j}$ and $\widetilde{F}^{nr}_{\overline{\omega}_j}$ denote complex basic pseudo-unramified extensions of k in one-to-one correspondence with the corresponding completions $F^{nr}_{\omega_j}$ and $F^{nr}_{\overline{\omega}_j}$ at the places ω_j and $\overline{\omega}_j$.

Infinite pseudo-ramified real places are assumed to be also characterized by Galois extension degrees: they are in fact classes of completions of which degrees are given by integers modulo N, $\mathbb{Z}/N\,\mathbb{Z}$, as follows:

$$[\widetilde{F}^+_{v_{j_\delta}}:k]=*+j\ N \qquad \text{and} \qquad [\widetilde{F}^+_{\overline{v}_{j_\delta}}:k]=*+j\ N$$

where:

- $F^+_{v_{j_\delta}}$ and $F^+_{\overline{v}_{j_\delta}}$ denote respectively a real basic ramified completion of F^+_L and of F^+_R in one-to-one correspondence with the splitting subsemifields $\widetilde{F}^+_{v_{j_\delta}}$ and $\widetilde{F}^+_{\overline{v}_{j_\delta}}$;
- $*$ denotes an integer inferior to N.

And, **infinite pseudo-ramified complex places** are similarly characterized by degrees given by the integers modulo N, $\mathbb{Z}/N\,\mathbb{Z}$, according to

$$[\widetilde{F}_{\omega_j}:k]=(*+j\ N)m^{(j)} \qquad \text{and} \qquad [\widetilde{F}_{\overline{\omega}_j}:k]=(*+j\ N)m^{(j)}$$

where:

- F_{ω_j} and $F_{\overline{\omega}_j}$ are respectively the basic complex pseudo-ramified completions of F_L and of F_R in one-to-one correspondence with the corresponding splitting subsemifields $\widetilde{F}_{\omega_j}$ and $\widetilde{F}_{\overline{\omega}_j}$;
- $m^{(j)}=\sup(m_{j_\delta}+1)$ is the **multiplicity of the $\boldsymbol{j_\delta}$-th real completion covering its $\boldsymbol{j}$-th complex equivalent** or the number of compactified divisors of F_{ω_j} and of $F_{\overline{\omega}_j}$.

- **The origin of the integer $\boldsymbol{N}$** in the real case results from the fact that the real pseudo-ramified completions $F^+_{v_{j_\delta}}$ and $F^+_{\overline{v}_{j_\delta}}$ are assumed to be generated respectively from the irreducible central subcompletions $F^+_{v^1_{j_\delta}}$ and $F^+_{\overline{v}^1_{j_\delta}}$ characterized by a (Galois extension) degree $[\widetilde{F}^+_{v^1_{j_\delta}}:k]=N$ and $[\widetilde{F}^+_{\overline{v}^1_{j_\delta}}:k]=N$.
  Similarly, the complex pseudo-ramified completions F_{ω_j} and $F_{\overline{\omega}_j}$ are generated respectively from equivalent subcompletions $F_{\omega^1_j}$ and $F_{\overline{\omega}^1_j}$ having a degree or rank equal to $N\centerdot m^{(j)}$.

- On the other hand, as **a place** is an equivalence class of completions, we have to consider a set of:
  - real pseudo-ramified (resp. pseudo-unramified) completions $\{F^{+(,nr)}_{v_{j_\delta,m_{j_\delta}}}\}_{m_{j_\delta}}$ and $\{F^{+(,nr)}_{\overline{v}_{j_\delta,m_{j_\delta}}}\}_{m_{j_\delta}}$, $1\le j_\delta\le r_\delta$, equivalent respectively to the corresponding basic

completions $F^{+(,nr)}_{v_{j_\delta}}$ and $F^{+(,nr)}_{\overline{v}_{j_\delta}}$, where $m_{j_\delta} \geq 1$ is an increasing integer such that $m^{(j_\delta)} = \sup(m_{j_\delta})$ denotes the multiplicity of $F^{+(,nr)}_{v_{j_\delta}}$ and of $F^{+(,nr)}_{\overline{v}_{j_\delta}}$;

- complex pseudo-ramified (resp. pseudo-unramified) completions $\{F^{(nr)}_{\omega_{j,m_j}}\}_{m_j}$ and $\{F^{(nr)}_{\overline{\omega}_{j,m_j}}\}_{m_j}$, $1 \leq j \leq r$, equivalent respectively to the corresponding basic completions $F^{(nr)}_{\omega_j}$ and $F^{(nr)}_{\overline{\omega}_j}$ where $m_j \geq 1$ is an increasing integer such that $m^{(j)}_\omega = \sup(m_j)$ refers to the multiplicity of $F^{(nr)}_{\omega_j}$ and $F^{(nr)}_{\overline{\omega}_j}$.

- All the **real pseudo-ramified completions** $F^+_{v_{j_\delta},m_{j_\delta}}$ (resp. $F^+_{\overline{v}_{j_\delta},m_{j_\delta}}$), $m_{j_\delta} \geq 1$, in a place v_{j_δ} (resp. $\overline{v}_{j_\delta}$), are characterized by the same (Galois extension) degree $\simeq j \centerdot N$ and **and are cut into j irreducible equivalent real subcompletions** $F_{v^{j'_\delta}_{j'_\delta}}$, $1 \leq j'_\delta \leq j_\delta$, having a degree equal to N.

  In the same manner, the complex pseudo-ramified completions $F_{\omega_{j,m_j}}$ (resp. $F_{\overline{\omega}_{j,m_j}}$), $m_j \geq 1$, in a place ω_j (resp. $\overline{\omega}_j$) are characterized by the same degree $\simeq j \centerdot m^{(j)} \centerdot N$ and are cut into j equivalent complex subcompletions $F_{\omega^{j'}_j}$, $1 \leq j' \leq j$, having a degree equal to $m^{(j)} \centerdot N$.

1.2 Definition: Infinite pseudo-ramified adele semirings and semigroups $F_{\omega_\oplus}$ and $F_{v_\oplus}$

- Infinite pseudo-ramified adele semirings $\mathbb{A}^\infty_{F_v}$, $\mathbb{A}^\infty_{F_{\overline{v}}}$, $\mathbb{A}^\infty_{F_\omega}$ and $\mathbb{A}^\infty_{F_{\overline{\omega}}}$ can be introduced by considering the products of the basic completions over primary places of respectively F^+_L, F^+_R, F_L and F_R according to:

$$\mathbb{A}^\infty_{F^+_v} = \prod_{j_{\delta_p}} F^+_{v_{j_{\delta_p}}} , \qquad \mathbb{A}^\infty_{F^+_{\overline{v}}} = \prod_{j_{\delta_p}} F^+_{\overline{v}_{j_{\delta_p}}} , \qquad 1 \leq j_{\delta_p} \leq r_\delta \leq \infty ,$$

$$\mathbb{A}^\infty_{F_\omega} = \prod_{j_p} F_{\omega_{j_p}} , \qquad \mathbb{A}^\infty_{F_{\overline{\omega}}} = \prod_{j_p} F_{\overline{\omega}_{j_p}} , \qquad 1 \leq j_p \leq r \leq \infty .$$

- Similarly, direct sums of completions will be given by:

$$F^+_{v_\oplus} = \bigoplus_{j_\delta} \bigoplus_{m_{j_\delta}} F^+_{v_{j_\delta},m_{j_\delta}} , \qquad F^+_{\overline{v}_\oplus} = \bigoplus_{j_\delta} \bigoplus_{m_{j_\delta}} F^+_{\overline{v}_{j_\delta},m_{j_\delta}} ,$$

$$F_{\omega_\oplus} = \bigoplus_{j} \bigoplus_{m_j} F_{\omega_{j,m_j}} , \qquad F_{\overline{\omega}_\oplus} = \bigoplus_{j} \bigoplus_{m_j} F_{\overline{\omega}_{j,m_j}} ,$$

1.3 Global inertia subgroups

- Let $\widetilde F_{\omega_{j,m_j}}$ (resp. $\widetilde F_{\overline\omega_{j,m_j}}$), $m_j > 1$, denote a complex pseudo-ramified extension corresponding to the respective pseudo-ramified completion $F_{\omega_{j,m_j}}$ (resp. $F_{\overline\omega_{j,m_j}}$) and approximatively equivalent to j basic complex pseudo-ramified extension $\widetilde F_{\omega_{j,1}}$ (resp. $\widetilde F_{\overline\omega_{j,1}}$), $m_j = 1$.

- Respectively, let $\{\widetilde F^{nr}_{\omega_{j,m_j}}\}_{m_j=1}^{m_j}$ (resp. $\{\widetilde F^{nr}_{\overline\omega_{j,m_j}}\}_{m_j=1}^{m_j}$) denote the set of complex pseudo-unramified extensions corresponding to the respective pseudo-unramified completions at the j-th pseudo-unramified complex place.

- Let $\operatorname{Gal}(\widetilde F^{nr}_{\omega_{j,m_j}}/k)$ (resp. $\operatorname{Gal}(\widetilde F^{nr}_{\overline\omega_{j,m_j}}/k)$) be the Galois subgroup of the pseudo-unramified complex extension $\widetilde F^{nr}_{\omega_{j,m_j}}$ (resp. $\widetilde F^{nr}_{\overline\omega_{j,m_j}}$) of k and let $\operatorname{Gal}(\widetilde F_{\omega_{j,m_j}}/k)$ (resp. $\operatorname{Gal}(\widetilde F_{\overline\omega_{j,m_j}}/k)$) be the Galois subgroup of the pseudo-ramified complex extension $\widetilde F_{\omega_{j,m_j}}$ (resp. $\widetilde F_{\overline\omega_{j,m_j}}$) of k.

- Then, the **global inertia subgroup** $I_{\widetilde F_{\omega_{j,m_j}}}$ (resp. $I_{\widetilde F_{\overline\omega_{j,m_j}}}$) of $\operatorname{Gal}(\widetilde F_{\omega_{j,m_j}}/k)$ (resp. $\operatorname{Gal}(\widetilde F_{\overline\omega_{j,m_j}}/k)$) will be defined by

$$\operatorname{Gal}(\widetilde F_{\omega_{j,m_j}}/k) = \operatorname{Gal}(\widetilde F^{nr}_{\omega_{j,m_j}}/k) \times I_{\widetilde F_{\omega_{j,m_j}}}$$
$$(\text{resp.}\quad \operatorname{Gal}(\widetilde F_{\overline\omega_{j,m_j}}/k) = \operatorname{Gal}(\widetilde F^{nr}_{\overline\omega_{j,m_j}}/k) \times I_{\widetilde F_{\overline\omega_{j,m_j}}}\ )$$

which leads to the exact sequence

$$1 \longrightarrow I_{\widetilde F_{\omega_{j,m_j}}} \longrightarrow \operatorname{Gal}(\widetilde F_{\omega_{j,m_j}}/k) \longrightarrow \operatorname{Gal}(\widetilde F^{nr}_{\omega_{j,m_j}}/k) \longrightarrow 1$$
$$(\text{resp.}\quad 1 \longrightarrow I_{\widetilde F_{\overline\omega_{j,m_j}}} \longrightarrow \operatorname{Gal}(\widetilde F_{\overline\omega_{j,m_j}}/k) \longrightarrow \operatorname{Gal}(\widetilde F^{nr}_{\overline\omega_{j,m_j}}/k) \longrightarrow 1\ ).$$

The global inertia subgroup $I_{\widetilde F_{\omega_{j,m_j}}}$ (resp. $I_{\widetilde F_{\overline\omega_{j,m_j}}}$) of order $N \centerdot m^{(j)}$ can then be considered as the subgroup of inner automorphisms of Galois while the Galois subgroup $\operatorname{Gal}(\widetilde F_{\omega_{j,m_j}}/k)$ (resp. $\operatorname{Gal}(\widetilde F_{\overline\omega_{j,m_j}}/k)$) can be viewed as a subgroup of modular automorphisms of Galois with respect to $I_{\widetilde F_{\omega_{j,m_j}}}$ (resp. $I_{\widetilde F_{\overline\omega_{j,m_j}}}$).

1.4 Shifted completions

- In the context of this paper, we have to introduce the shifted completions $F^{S_{\mathbb C}}_{\omega_{j,m_j}} = F_{\omega_{j,m_j}} \otimes \mathbb C$ (resp. $F^{S_{\mathbb C}}_{\overline\omega_{j,m_j}} = F_{\overline\omega_{j,m_j}} \otimes \mathbb C$) where $F_{\omega_{j,m_j}}$ (resp. $F_{\overline\omega_{j,m_j}}$) denotes the corresponding unshifted left (resp. right) complex pseudo-ramified completion.

  Notice that a set of shifted completions deals equivalently with

a) a difference ring [Coh] consisting in the ring of the unshifted completions and an isomorphism of this one onto the subring of shifted completions;

b) a $\mathrm{GL}_1(\mathbb{C})$-fibre bundle whose basis is the set of unshifted completions and total space the set of shifted completions: this case is the one-dimensional equivalent of the one envisaged in chapter 2 (for instance, see proposition 2.10).

So, the unshifted completions $F_{\omega_{j,m_j}}$ are in one-to-one correspondence with the pseudo-ramified extensions $\widetilde F_{\omega_{j,m_j}}$ of the symmetric splitting field $\widetilde F=\widetilde F_R\cup\widetilde F_L$ of the polynomial ring $k[x]$ while the shifted completions $F^{S_{\mathbb{C}}}_{\omega_{j,m_j}}$ are in one-to-one correspondence with pseudo-ramified extensions $\widetilde F^{S_{\mathbb{C}}}_{\omega_{j,m_j}}$ of the shifted symmetric splitting field $\widetilde F^{S_{\mathbb{C}}}=\widetilde F^{S_{\mathbb{C}}}_R\cup\widetilde F^{S_{\mathbb{C}}}_L$ of the difference polynomial subring $S(k[x])$.

- The **sum**, over j, of the set of **equivalent complex pseudo-ramified shifted completions** $F^{S_{\mathbb{C}}}_{\omega_{j,m_j}}$ (resp. $F^{S_{\mathbb{C}}}_{\overline\omega_{j,m_j}}$), $m_j\geq 1$ is given by:

$$F^{S_{\mathbb{C}}}_{\omega_\oplus}=\bigoplus_j\bigoplus_{m_j}F^{S_{\mathbb{C}}}_{\omega_{j,m_j}}\qquad\text{(resp.}\quad F^{S_{\mathbb{C}}}_{\overline\omega_\oplus}=\bigoplus_j\bigoplus_{m_j}F^{S_{\mathbb{C}}}_{\overline\omega_{j,m_j}}\ ).$$

1.5 Weil shifted global bilinear (semi)groups

- Let $\mathrm{Gal}(\widetilde F^{S_{\mathbb{C}}}_{\omega_j}/k)$ (resp. $\mathrm{Gal}(\widetilde F^{S_{\mathbb{C}}}_{\overline\omega_j}/k)$) denote the Galois subgroup of the shifted extension $\widetilde F^{S_{\mathbb{C}}}_{\omega_j}$ (resp. $\widetilde F^{S_{\mathbb{C}}}_{\overline\omega_j}$). Similarly, $\mathrm{Gal}(\widetilde F^{nr;S_{\mathbb{C}}}_{\omega_j}/k)$ (resp. $\mathrm{Gal}(\widetilde F^{nr;S_{\mathbb{C}}}_{\overline\omega_j}/k)$) will denote the Galois subgroup of the shifted pseudo-unramified extension $\widetilde F^{nr;S_{\mathbb{C}}}_{\omega_j}$ (resp. $\widetilde F^{nr;S_{\mathbb{C}}}_{\overline\omega_j}$).

- If $I_{\widetilde F^{S_{\mathbb{C}}}_{\omega_j}}$ (resp. $I_{\widetilde F^{S_{\mathbb{C}}}_{\overline\omega_j}}$) is the shifted global inertia subgroup of $\mathrm{Gal}(\widetilde F^{S_{\mathbb{C}}}_{\omega_j}/k)$ (resp. $\mathrm{Gal}(\widetilde F^{S_{\mathbb{C}}}_{\overline\omega_j}/k)$), then we have that:

$$\mathrm{Gal}(\widetilde F^{S_{\mathbb{C}}}_{\omega_j}/k)=\mathrm{Gal}(\widetilde F^{nr;S_{\mathbb{C}}}_{\omega_j}/k)\times I_{\widetilde F^{S_{\mathbb{C}}}_{\omega_j}}$$
$$\text{(resp.}\quad \mathrm{Gal}(\widetilde F^{S_{\mathbb{C}}}_{\overline\omega_j}/k)=\mathrm{Gal}(\widetilde F^{nr;S_{\mathbb{C}}}_{\overline\omega_j}/k)\times I_{\widetilde F^{S_{\mathbb{C}}}_{\overline\omega_j}}\ ).$$

$I_{\widetilde F^{S_{\mathbb{C}}}_{\omega_j}}$ (resp. $I_{\widetilde F^{S_{\mathbb{C}}}_{\overline\omega_j}}$) is the smallest normal subgroup (i.e. the subgroup of inner shifted automorphisms of Galois), of the subgroup $\mathrm{Gal}(\widetilde F^{S_{\mathbb{C}}}_{\omega_j}/k)$ (resp. $\mathrm{Gal}(\widetilde F^{S_{\mathbb{C}}}_{\overline\omega_j}/k)$) of modular shifted automorphisms of Galois.

- If it is assumed that the global Weil group $W^{ab}_{F^{S_{\mathbb{C}}}_{\omega}}$ (resp. $W^{ab}_{F^{S_{\mathbb{C}}}_{\overline\omega}}$) is the Galois subgroup referring to pseudo-ramified extensions characterized by extension degrees

$d=0 \mod N$, then we have that:

$$W^{ab}_{F^{S_{\mathbb{C}}}_{\omega}} \equiv \mathrm{Gal}(\dot{\widetilde{F}}^{S_{\mathbb{C}}}_{\omega_\oplus}/k) = \underset{j,m_j}{\oplus} \mathrm{Gal}(\dot{\widetilde{F}}^{S_{\mathbb{C}}}_{\omega_{j,m_j}}/k)$$

$$(\text{resp.}\quad W^{ab}_{F^{S_{\mathbb{C}}}_{\overline{\omega}}} \equiv \mathrm{Gal}(\dot{\widetilde{F}}^{S_{\mathbb{C}}}_{\overline{\omega}_\oplus}/k) = \underset{j,m_j}{\oplus} \mathrm{Gal}(\dot{\widetilde{F}}^{S_{\mathbb{C}}}_{\overline{\omega}_{j,m_j}}/k)\ ),$$

where $\dot{\widetilde{F}}^{S_{\mathbb{C}}}_{\omega_{j,m_j}}$ (resp. $\dot{\widetilde{F}}^{S_{\mathbb{C}}}_{\overline{\omega}_{j,m_j}}$) denote these shifted pseudo-ramified extensions with degrees $d=0 \mod N$.

This leads to the product of the **shifted global Weil groups** $W^{ab}_{F^{S_{\mathbb{C}}}_{\overline{\omega}_\oplus}} \times W^{ab}_{F^{S_{\mathbb{C}}}_{\omega_\oplus}}$

$$W^{ab}_{F^{S_{\mathbb{C}}}_{\overline{\omega}}} \times W^{ab}_{F^{S_{\mathbb{C}}}_{\omega}} = \mathrm{Gal}(\dot{\widetilde{F}}^{S_{\mathbb{C}}}_{\overline{\omega}_\oplus}/k) \times \mathrm{Gal}(\dot{\widetilde{F}}^{S_{\mathbb{C}}}_{\omega_\oplus}/k) \subset \mathrm{Gal}(\widetilde{F}^{S_{\mathbb{C}}}_{\overline{\omega}_\oplus}/k) \times \mathrm{Gal}(\widetilde{F}^{S_{\mathbb{C}}}_{\omega_\oplus}/k)\ .$$

1.6 From abelian class field theory to its nonabelian equivalence

The set of left (resp. right) pseudo-ramified extensions $\widetilde{F}_{\omega_{j,m_j}}$ (resp. $\widetilde{F}_{\overline{\omega}_{j,m_j}}$), $1 \le j \le r$, generates a one-dimensional complex affine semigroup $\mathbb{S}^1_L$ (resp. $\mathbb{S}^1_R$) in such a way the n-dimensional equivalent of their product $\mathbb{S}^1_R \times \mathbb{S}^1_L$ is a complex bilinear algebraic semigroup $G^{(2n)}(\widetilde{F}_{\overline{\omega}} \times \widetilde{F}_{\omega})$, isomorphic to the bilinear algebraic semigroup of matrices

$$\mathrm{GL}_n(\widetilde{F}_{\overline{\omega}} \times \widetilde{F}_{\omega}) = T^t_n(\widetilde{F}_{\overline{\omega}}) \times T_n(\widetilde{F}_{\omega})$$

where:

- $\widetilde{F}_{\omega} = \{\widetilde{F}_{\omega_1}, \dots, \widetilde{F}_{\omega_{j,m_j}}, \dots, \widetilde{F}_{\omega_{r,m_r}}\}$ (resp. $\widetilde{F}_{\overline{\omega}} = \{\widetilde{F}_{\overline{\omega}_1}, \dots, \widetilde{F}_{\overline{\omega}_{j,m_j}}, \dots, \widetilde{F}_{\overline{\omega}_{r,m_r}}\}$) denotes the set of complex pseudo-ramified finite extensions;
- $T_n(\widetilde{F}_{\omega})$ is the (semi)group of upper triangular matrices with entries in $\widetilde{F}_{\omega}$;
- $T^t_n(\widetilde{F}_{\overline{\omega}})$ is the (semi)group of lower triangular matrices with entries in $\widetilde{F}_{\overline{\omega}}$.

1.7 The algebraic general bilinear semigroup

- Let $\widetilde{B}_{F_\omega}$ (resp. $\widetilde{B}_{F_\omega}$) be a left (resp. right) **division semialgebra of complex dimension n** over the set $\widetilde{F}_{\omega}$ (resp. $\widetilde{F}_{\overline{\omega}}$) of the pseudo-ramified extensions $\widetilde{F}_{\omega_{j,m_j}}$ (resp. $\widetilde{F}_{\overline{\omega}_{j,m_j}}$) of k.

  Then, $\widetilde{B}_{F_\omega}$ (resp. $\widetilde{B}_{F_{\overline{\omega}}}$), which is a left (resp. right) vector semispace of complex dimension n over $\widetilde{F}_{\omega}$ (resp. $\widetilde{F}_{\overline{\omega}}$), is isomorphic to the algebra of Borel upper (resp. lower) triangular matrices:

$$\widetilde{B}_{F_\omega} \approx T_n(\widetilde{F}_{\omega}) \qquad (\text{resp.}\quad \widetilde{B}_{F_{\overline{\omega}}} \approx T^t_n(\widetilde{F}_{\overline{\omega}})\ )\ .$$

- This allows to define the **bilinear general semigroup** $\mathrm{GL}_n(\widetilde F_{\overline\omega}\times\widetilde F_\omega)$ by:

$$\widetilde B_{F_{\overline\omega}}\times\widetilde B_{F_{\overline\omega}}\simeq T^t_n(\widetilde F_{\overline\omega})\times T_n(\widetilde F_\omega)\equiv\mathrm{GL}_n(\widetilde F_{\overline\omega}\times\widetilde F_\omega)$$

  such that its representation space is given by the tensor product of a right $\widetilde B_{F_{\overline\omega}}$-semimodule $\widetilde M_R$ by a left $\widetilde B_{F_\omega}$-semimodule $\widetilde M_L$.

- Taking into account the definition of $\widetilde F_{\omega_\oplus}$ (resp. $\widetilde F_{\overline\omega_\oplus}$), the $\boldsymbol{\widetilde B_{F_{\omega_\oplus}}}$ **-semimodule** $\boldsymbol{\widetilde M_{L_\oplus}}$ (resp. $\widetilde B_{F_{\overline\omega_\oplus}}$-semimodule $\widetilde M_{R_\oplus}$) decomposes according to:

$$\widetilde M_{L_\oplus}=\bigoplus_j\bigoplus_{m_j}\widetilde M_{\omega_{j,m_j}}\qquad\left(\text{resp.}\quad\widetilde M_{R_\oplus}=\bigoplus_j\bigoplus_{m_j}\widetilde M_{\overline\omega_{j,m_j}}\ \right)$$

  where

$$\begin{aligned}\widetilde M_{\omega_{j,m_j}}&\simeq t_n(\widetilde F_{\omega_{j,m_j}})\subset T_n(\widetilde F_\omega)\\ \left(\text{resp.}\quad\widetilde M_{\overline\omega_{j,m_j}}\right.&\left.\simeq t^t_n(\widetilde F_{\overline\omega_{j,m_j}})\subset T^t_n(\widetilde F_{\overline\omega})\ \right)\end{aligned}$$

  is the representation subspace of $T_n(\widetilde F_\omega)$ (resp. $T^t_n(\widetilde F_{\overline\omega})$) restricted to the extension $\widetilde F_{\omega_{j,m_j}}$ (resp. $\widetilde F_{\overline\omega_{j,m_j}}$) and corresponds to the m_j-th representative of the j-th conjugacy class of $\widetilde M_L$ (resp. $\widetilde M_R$).

- Let $t_n(\widetilde F_{\omega_{j,m_j}})$ (resp. $t^t_n(\widetilde F_{\overline\omega_{j,m_j}})$) be an element of $T_n(\widetilde F_\omega)$ (resp. $T^t_n(\widetilde F_{\overline\omega})$) having the Levi decomposition:

$$\begin{aligned}t_n(\widetilde F_{\omega_{j,m_j}})&=d_n(\widetilde F_{\omega_{j,m_j}})\ u_n(\widetilde F_{\omega_{j,m_j}})\\ \left(\text{resp.}\quad t^t_n(\widetilde F_{\overline\omega_{j,m_j}})\right.&\left.=u^t_n(\widetilde F_{\overline\omega_{j,m_j}})\ d_n(\widetilde F_{\overline\omega_{j,m_j}})\ \right)\end{aligned}$$

  where $d_n(\centerdot)$ is a diagonal matrix of order n and where $u_n(\centerdot)$ is an upper unitriangular matrix.

  So, any matrix $g_n(\widetilde F_{\overline\omega_{j,m_j}}\times\widetilde F_{\omega_{j,m_j}})\in\mathrm{GL}_n(\widetilde F_{\overline\omega}\times\widetilde F_\omega)$ satisfies the **bilinear Gauss decomposition**:

$$g_n(\widetilde F_{\overline\omega_{j,m_j}}\times\widetilde F_{\omega_{j,m_j}})=[(d_n(\widetilde F_{\omega_{j,m_j}})\times d_n(\widetilde F_{\overline\omega_{j,m_j}}))][(u^t_n(\widetilde F_{\overline\omega_{j,m_j}})\ u_n(\widetilde F_{\omega_{j,m_j}}))]\ .$$

1.8 Pseudo-ramified lattices

Let $\mathcal O_{\widetilde F_\omega}$ (resp. $\mathcal O_{\widetilde F_{\overline\omega}}$) be the maximal order of $\widetilde F_\omega$ (resp. $\widetilde F_{\overline\omega}$). Then, $\Lambda_\omega=\mathcal O_{\widetilde B_{F_\omega}}$ (resp. $\Lambda_{\overline\omega}=\mathcal O_{\widetilde B_{F_{\overline\omega}}}$) in the division semialgebra $\widetilde B_{F_\omega}$ (resp. $\widetilde B_{F_{\overline\omega}}$) is a **pseudo-ramified**

$\mathbb{Z}\,/\boldsymbol{N}\ \mathbb{Z}$ **-lattice** in the left (resp. right) $\widetilde B_{F_\omega}$-semimodule $\widetilde M_L$ (resp. $\widetilde B_{F_{\overline\omega}}$-semimodule $\widetilde M_R$). So, we can fix the isomorphisms:

$$\Lambda_\omega \simeq T_n(\mathcal{O}_{\widetilde F_\omega}) \qquad \text{and} \qquad \Lambda_{\overline\omega} \simeq T_n^t(\mathcal{O}_{\widetilde F_{\overline\omega}})$$

leading to $\Lambda_{\overline\omega}\otimes\Lambda_\omega \simeq \mathrm{GL}_n(\mathcal{O}_{\widetilde F_{\overline\omega}}\times\mathcal{O}_{\widetilde F_\omega})$. And, if we take into account the decomposition of $\widetilde F_\omega$ and $\widetilde F_{\overline\omega}$ into their pseudo-ramified extensions, we have that the sublattice $\Lambda_{\omega_{j,m_j}}$ (resp. $\Lambda_{\overline\omega_{j,m_j}}$) into the $t_n(\widetilde F_{\omega_{j,m_j}})$-subsemimodule $\widetilde M_{\omega_{j,m_j}}$ (resp. $t_n^t(\widetilde F_{\overline\omega_{j,m_j}})$-subsemimodule $\widetilde M_{\overline\omega_{j,m_j}}$) verifies:

$$\Lambda_{\omega_{j,m_j}} \simeq t_n(\mathcal{O}_{\widetilde F_{\omega_{j,m_j}}}) \qquad (\text{resp.}\quad \Lambda_{\overline\omega_{j,m_j}} \simeq t_n^t(\mathcal{O}_{\widetilde F_{\overline\omega_{j,m_j}}})\ )$$

and

$$\Lambda_{\overline\omega_{j,m_j}}\otimes\Lambda_{\omega_{j,m_j}} \simeq g_n(\mathcal{O}_{\widetilde F_{\overline\omega,m_j}}\times\mathcal{O}_{\widetilde F_{\omega,m_j}}) \in \mathrm{GL}_n(\mathcal{O}_{\widetilde F_{\overline\omega}}\times\mathcal{O}_{\widetilde F_\omega})\ .$$

1.9 Proposition

Assume that we have fixed the isomorphism $\Lambda_{\overline\omega}\otimes\Lambda_\omega\simeq\mathrm{GL}_n((\mathbb{Z}\,/N\ \mathbb{Z}\,)^2)$.
Then, the representation space $\mathrm{Repsp}(\mathrm{GL}_n((\mathbb{Z}\,/N\ \mathbb{Z}\,)^2)$ *of* $\mathrm{GL}_n((\mathbb{Z}\,/N\ \mathbb{Z}\,)^2)$ *decomposes according to:*

$$\mathrm{Repsp}(\mathrm{GL}_n((\mathbb{Z}\,/N\ \mathbb{Z}\,)^2) = \bigoplus_j\bigoplus_{m_j}(\Lambda_{\overline\omega_{j,m_j}}\otimes\Lambda_{\omega_{j,m_j}})$$

where the direct sums bear over the places of F_ω *and* $F_{\overline\omega}$ *having multiplicities* $m^{(j)}=\sup(m_j)$.

1.10 Proposition

1) *The* **pseudo-ramified Hecke bialgebra** $\mathcal{H}_{R\times L}(n)$, *generated by all the pseudo-ramified Hecke bioperators* $T_R(n;t)\otimes T_L(n;t)$ *has a representation in the arithmetic subgroup of matrices* $\mathrm{GL}_n(\mathbb{Z}\,/N\ \mathbb{Z}\,)^2)$.
2) *The* **$\boldsymbol{j}$-th coset representative of $\boldsymbol{T_R(n;t)\otimes T_L(n;t)}$** *is given by:*

$$U_{j_R}\times U_{j_L} = [d_n(\mathcal{O}_{\widetilde F_{\overline\omega_{j,m_j}}})\centerdot d_n(\mathcal{O}_{\widetilde F_{\omega_{j,m_j}}})]\times[u_n^t(\mathcal{O}_{\widetilde F_{\overline\omega_{j,m_j}}})\centerdot u_n(\mathcal{O}_{\widetilde F_{\omega_{j,m_j}}})]\ .$$

3) *The Hecke bialgebra* $\mathcal{H}_{R\times L}(n)$ *generates the* **endomorphisms of the pseudo-ramified $\boldsymbol{\widetilde B_{F_{\overline\omega}}\otimes\widetilde B_{F_\omega}}$-bisemimodule** $\widetilde M_{R_\oplus}\otimes\widetilde M_{L_\oplus}$ *decomposing it according to the bisublattices* $(\Lambda_{\overline\omega_{j,m_j}}\otimes\Lambda_{\omega_{j,m_j}})$ *of* $\Lambda_{\overline\omega}\otimes\Lambda_\omega$:

$$\widetilde M_{R_\oplus}\otimes\widetilde M_{L_\oplus} = \bigoplus_j\bigoplus_{m_j}(\widetilde M_{\overline\omega_{j,m_j}}\otimes\widetilde M_{\omega_{j,m_j}})$$

where $(\widetilde M_{\overline\omega_{j,m_j}}\otimes\widetilde M_{\omega_{j,m_j}})$ *is a* $(\widetilde B_{F_{\overline\omega_j}}\times\widetilde B_{F_{\omega_j}})$*-bisubsemimodule representative.*

Proof. These assertions were proved in [Pie3]. ■

1.11 Corollary

There exists an injective morphism:

$$J_{\Lambda\to M}: \quad \Lambda_{\overline{\omega}}\otimes\Lambda_\omega \longrightarrow \widetilde{M}_R\otimes\widetilde{M}_L$$

from the bilattice $\Lambda_{\overline{\omega}}\otimes\Lambda_\omega$ *into the* $\mathrm{GL}_n(\widetilde{F}_{\overline{\omega}}\times\widetilde{F}_\omega)$ *-bisemimodule* $\widetilde{M}_R\otimes\widetilde{M}_L$ *.*

1.12 Toroidal compactification

- Let $\mathrm{GL}_n(\widetilde{F}_R\times\widetilde{F}_L)$ be the bilinear algebraic semigroup over the product of symmetric splitting semifields $\widetilde{F}_R$ and $\widetilde{F}_L$.

  Let $Y_{S_{R\times L}}=\mathrm{GL}_n(\widetilde{F}_R\times\widetilde{F}_L)\big/\mathrm{GL}_n((\mathbb{Z}/N\,\mathbb{Z})^2)$ be the non compact pseudo-ramified lattice bisemispace.

- The **Borel-Serre toroidal compactification of $\boldsymbol{Y_{S_{R\times L}}}$** is a toroidal projective emergent isomorphism of compactification given by:

  $$\gamma^c_{R\times L}: \qquad Y_{S_{R\times L}} \longrightarrow \overline{Y}_{S^T_{R\times L}}$$

  where:

  - $\overline{Y}_{S^T_{R\times L}}=\mathrm{GL}_n(F^T_R\times F^T_L)\big/\mathrm{GL}_n((\mathbb{Z}/N\,\mathbb{Z})^2)$;
  - F^T_R and F^T_L are toroidal compactifications of $\widetilde{F}_R$ and $\widetilde{F}_L$ respectively;

  such that:

  - $Y_{S_{R\times L}}$ may be viewed as the interior of $\overline{Y}_{S^T_{R\times L}}$ in the sense that the isomorphism $\gamma^c_{R\times L}$ is an inclusion isomorphism $Y_{S_{R\times L}}\hookrightarrow\overline{Y}_{S^T_{R\times L}}$ given by a homotopy equivalence;
  - $\overline{Y}_{S^T_{R\times L}}$ is a $\mathrm{GL}_n(F^T_{\overline{\omega}}\times F^T_\omega)$ -bisemimodule $M^T_R\otimes M^T_L$ over the sets $F^T_\omega=\{F^T_{\omega_1},\dots,F^T_{\omega_{r,mr}}\}$ and $F^T_{\overline{\omega}}=\{F^T_{\overline{\omega}_1},\dots,F^T_{\overline{\omega}_{r,mr}}\}$ of toroidal completions.

  By this way, $\gamma^c_{R\times L}$ sends all equivalent representatives of conjugacy classes of $\mathrm{GL}_n(\widetilde{F}_R\times\widetilde{F}_L)$ into their toroidal compactified equivalents which are products of n-dimensional complex semitori $T^{2n}_R[j,m_j]\times T^{2n}_L[j,m_j]$.

- On the other hand, let $F^T_{\omega^1}=\{F^T_{\omega^1_1},\dots,F^T_{\omega_{j,mr}}\}$ (resp. $F^T_{\overline{\omega}^1}=\{F^T_{\overline{\omega}^1_1},\dots,F^T_{\overline{\omega}_{j,mr}}\}$) denote the set of irreducible toroidal completions.

  The **bilinear complex parabolic semigroup $\boldsymbol{P_n(F^T_{\overline{\omega}^1_1}\times F^T_{\omega^1})}$** is the smallest normal bilinear subsemigroup of $\mathrm{GL}_n(F^T_{\overline{\omega}}\times F^T_{\omega})$, representing the n-fold product of the global inertia subgroup $I_{\widetilde F_{\overline{\omega}}}\times I_{\widetilde F_{\omega}}$.

  The **double coset decomposition** of $\mathrm{GL}_n(F^T_R\times F^T_L)$ gives rise to the compactified bisemispace [Vog]:

$$S^{P_n}_{\mathrm{GL}_n}=P_n(F^T_{\overline{\omega}^1}\times F^T_{\omega^1})\setminus \mathrm{GL}_n(F^T_R\times F^T_L)/\,\mathrm{GL}_n((\mathbb{Z}/N\,\mathbb{Z})^2)\ .$$

1.13 Proposition

As a consequence of the double coset decomposition of the compactified bisemivariety $\overline{S}^{P_n}_{K_n}$, *the* **modular conjugacy classes of $\boldsymbol{\mathrm{GL}_n(F^T_{\overline{\omega}}\times F^T_{\omega})}$** *with respect to the bilinear parabolic semigroup* $P_n(F^T_{\overline{\omega}^1}\times F^T_{\omega^1})$ *correspond to the cosets of the compactified pseudo-ramified lattice bisemispace* $\overline{Y}_{S^T_{R\times L}}=\mathrm{GL}_n(F^T_R\times F^T_L)/\,\mathrm{GL}_n((\mathbb{Z}/N\,\mathbb{Z})^2)$.

Proof. As the bilinear parabolic semigroup $P_n(F^T_{\overline{\omega}^1}\times F^T_{\omega^1})$ is compact and as the cosets of $\overline{Y}_{S^T_{R\times L}}$ correspond to the set of lattices of $((F^T_{\overline{\omega}})^n\times(F^T_{\omega})^n)$, we have that:

$$P_n(F^T_{\overline{\omega}^1}\times F^T_{\omega^1})/\,\mathrm{GL}_n(F^T_R\times F^T_L)\approx \mathrm{GL}_n(F^T_R\times F^T_L)/\,\mathrm{GL}_n((\mathbb{Z}/N\,\mathbb{Z})^2)$$

implying that the modular conjugacy classes of $\mathrm{GL}_n(F^T_{\overline{\omega}}\times F^T_{\omega})$ are the cosets of the bilinear quotient semigroup $P_n(F^T_{\overline{\omega}^1}\times F^T_{\omega^1})/\,\mathrm{GL}_n(F^T_R\times F^T_L)$. ■

1.14 Reducible Galois cohomologies

- Let $n=n_1+\cdots+n_s$ be a partition of n [Rod], [Zel] and let

$$\begin{aligned}\overline{Y}^{2n=2n_1+\cdots+2n_s}_{S_{R\times L}}&=\mathrm{GL}_{n=n_1+\cdots+n_s}(F_R\times F_L)/\,\mathrm{GL}_n((\mathbb{Z}/N\,\mathbb{Z})^2)\\&=\mathrm{Repsp}(\mathrm{GL}_{n=n_1+\cdots+n_s}(F_{\overline{\omega}}\times F_\omega))=\boxplus_{n_\ell}\mathrm{Repsp}(\mathrm{GL}_{n_\ell}(F_{\overline{\omega}}\times F_\omega))\end{aligned}$$

  be the **reducible compactified representation space of** $\mathrm{GL}_n(\widetilde F_{\overline{\omega}}\times\widetilde F_{\omega})$ decomposing according to the irreducible representation spaces $\mathrm{Repsp}(\mathrm{GL}_{n_\ell}(F_{\overline{\omega}}\times F_\omega))$ of $\mathrm{GL}_n(F_{\overline{\omega}}\times F_\omega)$ given with respect to modular conjugacy classes "j".

- **The bilinear cohomology of** $\overline{Y}^{2n=2n_1+\cdots+2n_s}_{S_{R\times L}}$, introduced in [Pie3], (section 3.2), decomposes according to
$$H^*(\overline{Y}^{2n=2n_1+\cdots+2n_s}_{S_{R\times L}}, M_R^{2n}\otimes M_L^{2n}) = \bigoplus_{2n_\ell=2n_1}^{2n_s} H^{2n_\ell}(\overline{Y}^{2n}_{S_{R\times L}}, M_R^{2n_\ell}\otimes M_L^{2n_\ell})$$
where $M_L^{2n_\ell}$ (resp. $M_R^{2n_\ell}$) is a left (resp. right) $T_{n_\ell}(F_\omega)$-subsemimodule (resp. $T^t_{n_\ell}(F_{\overline{\omega}})$-subsemimodule) of real dimension $2n_\ell$.
As $M^{2n}\otimes M^{2n}\equiv G^{(2n)}(F_{\overline{\omega}}\times F_\omega)$ is a smooth abstract bisemivariety, the bilinear cohomology on its algebraic equivalent $\widetilde{M}^{2n}\otimes\widetilde{M}^{2n}\equiv G^{(2n)}(\widetilde{F}_{\overline{\omega}}\times\widetilde{F}_\omega)$ will be similarly decomposed into
$$H^*(G^{(2n)}(\widetilde{F}_{\overline{\omega}}\times\widetilde{F}_\omega)) = \bigoplus_{2n_\ell} H^{2n_\ell}(G^{(2n)}(\widetilde{F}_{\overline{\omega}}\times\widetilde{F}_\omega), \widetilde{M}_R^{2n_\ell}\otimes\widetilde{M}_L^{2n_\ell}) \; .$$

- The cohomology of the reducible toroidal bisemivariety $\overline{S}^{P_n}_{\mathrm{GL}_n}$ also decomposes according to:
$$H^*(\overline{S}^{P_n=n_1+\cdots+n_s}_{\mathrm{GL}_n=n_1+\cdots+n_s}, M_{T_R}^{2n}\otimes M_{T_L}^{2n}) = \bigoplus_{2n_\ell} H^{2n_\ell}(\overline{S}^{P_n}_{\mathrm{GL}_n}, M_{T_R}^{2n_\ell}\otimes M_{T_L}^{2n_\ell})$$
where $M_{T_L}^{2n_\ell}$ (resp. $M_{T_R}^{2n_\ell}$) is a left (resp. right) compactified $T_{n_\ell}(F_\omega^T)$ (resp. $T^t_{n_\ell}(F^T_{\overline{\omega}})$)-subsemimodule of dimension $2n_\ell$.

- However, **the coefficients of the cohomology** are generally considered in (bisemi)-sheaves of rings over bilinear complete algebraic semigroups $(M_R^{2n_\ell}\otimes M_L^{2n_\ell})$.
In this purpose, **a (semi)sheaf $\widehat{M}_L^{2n_\ell}$ (resp. $\widehat{M}_R^{2n_\ell}$) of C^∞-differentiable functions on $M_L^{2n_\ell}$ (resp. $M_R^{2n_\ell}$)** will be envisaged and a (bisemi)sheaf $(\widehat{M}_R^{2n_\ell}\otimes\widehat{M}_L^{2n_\ell})$ of C^∞-differentiable bifunctions (i.e. products of cofunctions by functions) on $(M_R^{2n_\ell}\otimes M_L^{2n_\ell})$ will be considered as coefficients of the cohomology $H^{2n_\ell}(\overline{Y}^{2n}_{S_{R\times L}}, \widehat{M}_R^{2n_\ell}\otimes\widehat{M}_L^{2n_\ell})$.

1.15 Algebraic semicycles of the Chow (semi)groups

Let Y_L (resp. Y_R) denote a left (resp. right) algebraic semigroup $G^{(2n)}(F_\omega)$ (resp. $G^{(2n)}(F_{\overline{\omega}})$) of complex dimension n isomorphic to a left (resp. right) smoth semischeme. Then, the algebraic semicycle $\mathrm{CY}^{2n_\ell}(Y_L)$ (resp. $\mathrm{CY}^{2n_\ell}(Y_R)$) of dimension $2n_\ell$ on Y_L (resp. Y_R) is such that:
$$\mathrm{CY}^{2n_\ell}(Y_L)\subset \mathcal{Z}^{2n_\ell}(Y_L)\subset \mathrm{CH}^{2n_\ell}(Y_L)$$
$$(\text{resp.}\quad \mathrm{CY}^{2n_\ell}(Y_R)\subset \mathcal{Z}^{2n_\ell}(Y_R)\subset \mathrm{CH}^{2n_\ell}(Y_R)\ )$$

where:

- $\not{Z}^{2n_\ell}(Y_L)$ is the **semigroup of algebraic semicycles** $\mathrm{CY}^{2n_\ell}(Y_L)$ of codimension $2n_\ell$;
- $\mathrm{CH}^{2n_\ell}(Y_L)=\dfrac{\not{Z}^{2n_\ell}(Y_L)}{\not{Z}^{2n_\ell}_{\mathrm{rat}}(Y_L)}$ is the **$\boldsymbol{2n_\ell}$-th Chow semigroup of $\boldsymbol{Y_L}$** with $\not{Z}^{2n_\ell}_{\mathrm{rat}}(Y_L)$ the semigroup of algebraic semicycles of codimension $2n_\ell$ rationally equivalent to zero [Mur], [Vis].

It is evident that $\mathrm{CY}^{2n_\ell}(Y_L)$ (resp. $\mathrm{CY}^{2n_\ell}(Y_R)$) decomposes according to the equivalence classes " j " having representatives m_j such that:

$$\mathrm{CY}^{2n_\ell}(Y_R)\times\mathrm{CY}^{2n_\ell}(Y_L)=\bigoplus_j\bigoplus_{m_j}(\mathrm{CY}^{2n_\ell}(Y_R[j,m_j])\times\mathrm{CY}^{2n_\ell}(Y_L[j,m_j])\ .$$

1.16 Suslin-Voevodsky motivic presheaf [Frie]

- Let $\Sigma_L^{2n_\ell}$ (resp. $\Sigma_R^{2n_\ell}$) denote a left (resp. right) complex topological $2n_\ell$-simplex and let $\Sigma_L^{\boldsymbol{\cdot}}$ (resp. $\Sigma_R^{\boldsymbol{\cdot}}$) denote a cosimplicial object from the collection of the $\Sigma_L^{2n_\ell}$ (resp. $\Sigma_R^{2n_\ell}$) in the category $Sm_L(k)$ (resp. $Sm_R(k)$) of left (resp. right) (semi)schemes over k .

- A **Suslin-Voevodsky motivic left (resp. right) presheaf** of the left (resp. right) (semi)scheme X_L^{sv} (resp. X_R^{sv}) of complex dimension ℓ on $Sm_L(k)$ (resp. $Sm_R(k)$) and denoted $\underline{c}_*(X_L^{\mathrm{sv}})$ (resp. $\underline{c}_*(X_R^{\mathrm{sv}})$) is a functor from X_L^{sv} (resp. X_R^{sv}) to the left (resp. right) chain complex associated to the abelian semigroup $\bigsqcup\limits_{i_\ell}\mathrm{Hom}_{Sm_L(k)}(\dot{\Sigma}_L,SP^{i_\ell}(X_L^{\mathrm{sv}}))$ (resp. $\bigsqcup\limits_{i_\ell}\mathrm{Hom}_{Sm_R(k)}(\dot{\Sigma}_R,SP^{i_\ell}(X_R^{\mathrm{sv}}))$) where $SP^{i_\ell}(X_L^{\mathrm{sv}})$ (resp. $SP^{i_\ell}(X_R^{\mathrm{sv}})$) denotes the i_ℓ-th symmetric product of X_L^{sv} (resp. X_R^{sv}).

- On the other hand, let $\boldsymbol{Z_L(2n_\ell)}$ **(resp.** $\boldsymbol{Z_R(2n_\ell)}$ **)** denote the left (resp. right) **Suslin-Voevodsky submotive of dimension $\boldsymbol{2n_\ell=i_\ell\times 2\ell}$** as developed in chapter 1 of [Pie3]. $Z_L(2n_\ell)$ (resp. $Z_R(2n_\ell)$) can be checked to correspond to a left (resp. right) element of the $2n_\ell$-th semigroup $\not{Z}^{2n_\ell}(Y_L)$ (resp. $\not{Z}^{2n_\ell}(Y_R)$) of left (resp. right) algebraic semicycles over Y_L (resp. Y_R) of dimension $2n$.

- Similarly, let $Z_L^T(2n_\ell)$ (resp. $Z_R^T(2n_\ell)$) be the resulting toroidal compactified Suslin-Voevodsky submotive obtained from $Z_L(2n_\ell)$ (resp. $Z_R(2n_\ell)$).

1.17 Proposition

The cohomologies $H^{2n_\ell}(Y_{R\times L},\mathrm{CY}^{2n_\ell}(Y_R)\times\mathrm{CY}^{2n_\ell}(Y_L))$ *and* $H^{2n_\ell}(\underline{c}_*(X^{\rm sv}_{R\times L}),Z_{R\times L}(2n_\ell))$ *are* ***bilinear pure motivic cohomologies****, with* $Y_{R\times L}=Y_R\times Y_L$ *and* $\underline{c}_*(X^{\rm sv}_{R\times L})=\underline{c}_*(X^{\rm sv}_R)\times\underline{c}_*(X^{\rm sv}_L)$ *.*

Proof. Let $Z_{R\times L}(2n_\ell)\equiv Z_R(2n_\ell)\otimes Z_L(2n_\ell)$ denote the bilinear products [Pie5], right by left, of Suslin-Voevodsky submotives of complex codimension n_ℓ.
If we have the isomorphisms:

$$\begin{aligned} i_{M-X}: \quad & H^{2n_\ell}(Y_{R\times L},\mathrm{CY}^{2n_\ell}(Y_R)\times\mathrm{CY}^{2n_\ell}(Y_L)) \\ & \qquad \xrightarrow{\sim} H^{2n_\ell}(\underline{c}_*(X^{\rm sv}_{R\times L}),Z_{R\times L}(2n_\ell))) \end{aligned}$$

resulting from sections 1.15 and 1.16., and [Pie3],
it is evident that the cohomologies

$$H^{2n_\ell}(Y_{R\times L},\mathrm{CY}^{2n_\ell}(Y_R)\times\mathrm{CY}^{2n_\ell}(Y_L))=\mathrm{Hom}_{C_{M_{R\times L}}}(Y_{R\times L},\mathrm{CY}^{2n_\ell}(Y_R)\times\mathrm{CY}^{2n_\ell}(Y_L))$$

and $H^{2n_\ell}(\underline{c}_*(X^{\rm sv}_{T_{R\times L}},Z_{R\times L}(2n_\ell))$ are "pure" motivic, noticing that $C_{M_{R\times L}}$ is the category of smooth bisemischemes isomorphic to $\mathrm{GL}_{n_\ell}(F_{\overline{\omega}}\times F_\omega)$-bisemimodules $M_R^{2n_\ell}\otimes M_L^{2n_\ell}$. ■

2 Bilinear cohomologies of mixed (bisemi)motives

The objective consists now in introducing a left and a right triangulated category $DM_L(k)$ and $DM_R(k)$ of mixed (semi)motives [Del2], [Del3], [Jan2] and in developing a corresponding suitable bilinear mixed motivic cohomology.

2.1 Definition: Correspondences on Suslin-Voevodsky (semi)-motives

The Suslin-Voevodsky left (resp. right) semimotive $\underline{c}_*(X_L^{\rm sv})$ (resp. $\underline{c}_*(X_R^{\rm sv})$), also noted $M(X_L^{\rm sv})$ (resp. $M(X_R^{\rm sv})$) has the property to be a left (resp. right) **presheaf with transfers** [Mor], [Frie]. That is to say that there exist **left (resp. right) correspondences**, noted $\mathrm{Corr}(SP^{f_\ell}(X_L^{\rm sv}),X_L^{2n_\ell-2f_\ell\cdot\ell})$ (resp. $\mathrm{Corr}(SP^{f_\ell}(X_R^{\rm sv}),X_R^{2n_\ell-2f_\ell\cdot\ell})$), on the set of irreducible subvarieties of $X_L^{2n_\ell}$ (resp. $X_R^{2n_\ell}$).
Left (resp. right) correspondences are here introduced by:

$$\mathrm{Corr}(SP^{f_\ell}(X_L^{\rm sv}),X_L^{2n_\ell-2f_\ell\cdot\ell}):\ SP^{i_\ell}(X_L^{\rm sv})\to X_L^{2n_\ell}=SP^{f_\ell}(X_L^{\rm sv})\times X_L^{2n_\ell-2f_\ell\cdot\ell}$$

$$(\text{resp.}\quad \mathrm{Corr}(SP^{f_\ell}(X_R^{\rm sv}),X_R^{2n_\ell-2f_\ell\cdot\ell}):\ SP^{i_\ell}(X_R^{\rm sv})\to X_R^{2n_\ell}=SP^{f_\ell}(X_R^{\rm sv})\times X_R^{2n_\ell-2f_\ell\cdot\ell}\ ),$$

for the integers

- $f_\ell\cdot\ell\le n_\ell\le n\,$;
- $i_\ell\times\ell=n_\ell=(f_\ell\cdot\ell)+(n_\ell-f_\ell\cdot\ell)\,$;
- $f_\ell\le i_\ell\,$;

such that:

- the i_ℓ-th sub(semi)motive $SP^{i_\ell}(X_L^{\rm sv})$ of dimension $2n_\ell=i_\ell\times 2\ell$ be sent by the left correspondence $\mathrm{Corr}(\cdot,\cdot)$ to the product $X_L^{2n_\ell}$ of closed irreducible sub(semi)motives $SP^{f_\ell}(X_L^{\rm sv})$ by $X_L^{2n_\ell-2f_\ell\cdot\ell}$ where $X_L^{2n_\ell-2f_\ell\cdot\ell}$ is a smooth presheaf of complex dimension $n_\ell-f_\ell\cdot\ell\,$;
- there exists a projection from $X_L^{2n_\ell}$ (resp. $X_R^{2n_\ell}$) to an irreducible component of $SP^{f_\ell}(X_L^{\rm sv})$ (resp. $SP^{f_\ell}(X_R^{\rm sv})$).

2.2 Definition: Fibre of the tangent bundle $\mathrm{Tan}(SP^{f_\ell}(X^{\mathrm{sv}}_{L,R}))$

Let $\mathrm{TAN}[SP^{f_\ell}(X^{\mathrm{sv}}_L)]$ (resp. $\mathrm{TAN}[SP^{f_\ell}(X^{\mathrm{sv}}_R)]$) be the left (resp. right) tangent vector bundle given by the triple:

$$\begin{aligned} &\mathrm{Tan}[SP^{f_\ell}(X^{\mathrm{sv}}_L)](\Delta^{2f_\ell\cdot\ell}_L, \mathrm{pr}_L, SP^{f_\ell}(X^{\mathrm{sv}}_L)) \\ (\text{resp.} \quad &\mathrm{Tan}[SP^{f_\ell}(X^{\mathrm{sv}}_R)](\Delta^{2f_\ell\cdot\ell}_R, \mathrm{pr}_R, SP^{f_\ell}(X^{\mathrm{sv}}_R)) \ ) \end{aligned}$$

where:

- $\Delta^{2f_\ell\cdot\ell}_L$ (resp. $\Delta^{2f_\ell\cdot\ell}_R$) is the total space obtained from the base space $SP^{f_\ell}(X^{\mathrm{sv}}_L)$ (resp. $SP^{f_\ell}(X^{\mathrm{sv}}_R)$) under the action of the upper (resp. lower) linear trigonal group $T_{f_\ell\cdot\ell}(\mathbb{C})$ (resp. $T^t_{f_\ell\cdot\ell}(\mathbb{C})$) $\subset \mathrm{GL}_{f_\ell\cdot\ell}(\mathbb{C}\times\mathbb{C})$ such that

$$\Delta^{2f_\ell\cdot\ell}_L = SP^{f_\ell}(X^{\mathrm{sv}}_L) \times \mathrm{AdFRepsp}(T_{f_\ell\cdot\ell}(\mathbb{C}))$$

$$(\text{resp.} \quad \Delta^{2f_\ell\cdot\ell}_R = SP^{f_\ell}(X^{\mathrm{sv}}_R) \times \mathrm{AdFRepsp}(T_{f_\ell\cdot\ell}(\mathbb{C})) \ )$$

  be defined with respect to the **left (resp. right) fibre $\mathrm{AdFRepsp}(T_{f_\ell\cdot\ell}(\mathbb{C}))$ (resp. $\mathrm{AdFRepsp}(T^t_{f_\ell\cdot\ell}(\mathbb{C}))$) which is given by the adjoint functional representation space of $T_{f_\ell\cdot\ell}(\mathbb{C})$ (resp. $T^t_{f_\ell\cdot\ell}(\mathbb{C})$);**

- pr_L (resp. pr_R) is the evident projection:

$$\mathrm{pr}_L : \quad \Delta^{2f_\ell\cdot\ell}_L \longrightarrow SP^{f_\ell}(X^{\mathrm{sv}}_L) \qquad (\text{resp.} \quad \mathrm{pr}_R : \quad \Delta^{2f_\ell\cdot\ell}_R \longrightarrow SP^{f_\ell}(X^{\mathrm{sv}}_R) \ ) \ .$$

2.3 Definition: Shifted correspondences

Taking into account the left (resp. right) tangent bundle $\mathrm{TAN}[SP^{f_\ell}(X^{\mathrm{sv}}_L)]$ (resp. $\mathrm{TAN}[SP^{f_\ell}(X^{\mathrm{sv}}_R)]$) as introduced in definition 2.2. and the left (resp. right) correspondences $\mathrm{Corr}(SP^{f_\ell}(X^{\mathrm{sv}}_L), X^{2n_\ell-2f_\ell\cdot\ell}_L)$ (resp. $\mathrm{Corr}(SP^{f_\ell}(X^{\mathrm{sv}}_R), X^{2n_\ell-2f_\ell\cdot\ell}_R)$) on Suslin-Voevodsky semimotives, **shifted left (resp. right) correspondences can be defined by the homomorphism**:

$$\mathrm{CORR}^S_L : \quad \mathrm{Corr}(SP^{f_\ell}(X^{\mathrm{sv}}_L), X^{2n_\ell-2f_\ell\cdot\ell}_L) \longrightarrow \mathrm{Corr}^S_L(\Delta^{2f_\ell\cdot\ell}_L, X^{2n_\ell-2f_\ell\cdot\ell}_L)$$

$$(\text{resp.} \quad \mathrm{CORR}^S_L : \quad \mathrm{Corr}(SP^{f_\ell}(X^{\mathrm{sv}}_R), X^{2n_\ell-2f_\ell\cdot\ell}_R) \longrightarrow \mathrm{Corr}^S_R(\Delta^{2f_\ell\cdot\ell}_L, X^{2n_\ell-2f_\ell\cdot\ell}_R) \ )$$

where the left (resp. right) smooth presheaf $SP^{f_\ell}(X^{\mathrm{sv}}_L)$ (resp. $SP^{f_\ell}(X^{\mathrm{sv}}_R)$) has been sent to the corresponding smooth presheaf

$$\Delta^{2f_\ell\cdot\ell}_L = SP^{f_\ell}(X^{\mathrm{sv}}_L) \times \mathrm{AdFRepsp}(T_{f_\ell\cdot\ell}(\mathbb{C}))$$

$$(\text{resp.} \quad 2\Delta^{f_\ell\cdot\ell}_R = SP^{f_\ell}(X^{\mathrm{sv}}_R) \times \mathrm{AdFRepsp}(T^t_{f_\ell\cdot\ell}(\mathbb{C})) \ )$$

by means of the inverse projection map pr_L^{-1} (resp. pr_R^{-1}) of $\mathrm{Tan}[SP^{f_\ell}(X_L^{\mathrm{sv}})]$ (resp. $\mathrm{Tan}[SP^{f_\ell}(X_R^{\mathrm{sv}})]$).

2.4 Triangulated category of mixed (semi)motives

- Let the Suslin-Voevodsky left (resp. right) pure (semi)motive $M(X_L^{\mathrm{sv}})$ (resp. $M(X_R^{\mathrm{sv}})$), provided with left (resp. right) shifted correspondences $\mathrm{Corr}^S(\centerdot,\centerdot)$, be noted $M_{DM_L}(X_L^{\mathrm{sv}})$ (resp. $M_{DM_R}(X_R^{\mathrm{sv}})$): it is then a **left (resp. right) mixed (semi)motive** of the triangulated category $DM_L(k)$ (resp. $DM_R(k)$) of left (resp. right) geometric (semi)motives. Indeed, the isomorphism:

$$
\begin{aligned}
M_{\mathrm{Corr}_L}: \qquad & M(X_L^{\mathrm{sv}}) = \bigsqcup_{i_\ell} \mathrm{Hom}_{Sm_L(k)}(\Sigma^{\centerdot}_L, SP^{i_\ell}(X_L^{\mathrm{sv}})) \\
& \qquad \longrightarrow M_{DM_L}(X_L^{\mathrm{sv}}) = \bigsqcup_{i_\ell,f_\ell} \mathrm{Hom}_{Sm_L(k)}(SP^{i_\ell}(X_L^{\mathrm{sv}}), X_L^{2n_\ell}[2f_\ell\cdot\ell]),
\end{aligned}
$$

  where $X_L^{2n_\ell}[f_\ell\cdot\ell] = \Delta_L^{2f_\ell\cdot\ell}\times X^{2n_\ell-2f_\ell\cdot\ell}$, maps the Suslin-Voevodsky pure (semi)motive $M(X_L^{\mathrm{sv}})$ to the Suslin-Voevodsky mixed (semi)motive $M_{DM_L}(X_L^{\mathrm{sv}})$ by means of the left shifted correspondence $\mathrm{Corr}^S(\Delta_L^{2f_\ell\cdot\ell}, X_L^{2n_\ell-2f_\ell\cdot\ell})$, taking into account that $\Delta_L^{2f_\ell\cdot\ell}$ **is a sub(semi)motive shifted in $2f_\ell\cdot\ell$-dimensions**.

- Noticing that a triangulated category is an additive category graded by a translation functor and a set of distinguished triangles [Ver], we have that the isomorphism M_{Corr_L} can be viewed as belonging to the translation functor from the category of Suslin-Voevodsky pure (semi)motives to the **triangulated category $DM_L(k)$ of mixed (semi)motives** [Hub], [C-F].

  And, the derived category $D(M(X_L^{\mathrm{sv}}))$ (resp. $D(M(X_R^{\mathrm{sv}}))$) of pure left (resp. right) (semi)motives $M(X_L^{\mathrm{sv}})$ (resp. $M(X_R^{\mathrm{sv}})$) with transfers is included into the corresponding triangulated category $DM_L(k)$ (resp. $DM_R(k)$), **a derived category** resulting from a corresponding triangulated category with a condition of null homotopy on the automorphisms of translations [F-S-V].

- **Remark** finally that a triangulated category of mixed (semi)motives can also be defined from the toroidal pure (semi)motives $\underline{c}_*(X_{T_L}^{\mathrm{sv}})$ (resp. $\underline{c}_*(X_{T_R}^{\mathrm{sv}})$): so, **the Suslin-Voevodsky left (resp. right) mixed (semi)motives** $M_{DM_L}(X_{T_L}^{\mathrm{sv}})$ (resp. $M_{DM_R}(X_{T_R}^{\mathrm{sv}})$) belong to the left (resp. right) derived category $D(M(X_{T_L}^{\mathrm{sv}}))$ (resp. $D(M(X_{T_R}^{\mathrm{sv}}))$).

2.5 Lemma

Let

$$\Delta_L^{2f_\ell\cdot\ell} = SP^{f_\ell}(X_L^{\rm sv}) \times \mathrm{AdFRepsp}(T_{f_\ell\cdot\ell}(\mathbb{C}\,))$$

$$\textit{(resp.}\quad \Delta_R^{2f_\ell\cdot\ell} = SP^{f_\ell}(X_R^{\rm sv}) \times \mathrm{AdFRepsp}(T^t_{f_\ell\cdot\ell}(\mathbb{C}\,))\ )$$

be a left (resp. right) $2f_\ell\cdot\ell$ *-(semi)scheme in the category* $Sm_L(k)$ *(resp.* $Sm_R(k)$ *) of left (resp. right) smooth (semi)schemes over* k *.*
Then, we have that the vector bisemispace $\mathrm{Aut}(\mathrm{Tan}_e(SP^{f_\ell}(X_R^{\rm sv}) \times SP^{f_\ell}(X_L^{\rm sv})))$ *of the endomorphisms of the tangent bisemispace of* $SP^{f_\ell}(X_R^{\rm sv})\times SP(X_L^{\rm sv})$ *is precisely a* $(f_\ell\cdot\ell\times f_\ell\cdot\ell)$ *-subbisemischeme of* $\Delta_R^{f_\ell\cdot\ell}\times\Delta_L^{f_\ell\cdot\ell}$ *such that:*

$$\begin{aligned}\Delta_R^{2f_\ell\cdot\ell}\times\Delta_L^{2f_\ell\cdot\ell} &\simeq \mathrm{FRepsp}(\mathrm{GL}_{f_\ell\cdot\ell}((F_{\overline{\omega}}\otimes\mathbb{C}\,)\times(F_\omega\otimes\mathbb{C}\,)))\\ &= \mathrm{AdFRepsp}(\mathrm{GL}_{f_\ell\cdot\ell}(\mathbb{C}\,\times\mathbb{C}\,))\times\mathrm{FRepsp}(\mathrm{GL}_{f_\ell\cdot\ell}(F_{\overline{\omega}}\times F_\omega))\end{aligned}$$

where:

- $SP^{f_\ell}(X_R^{\rm sv})\times SP^{f_\ell}(X_L^{\rm sv}) \simeq \mathrm{FRepsp}(\mathrm{GL}_{f_\ell\cdot\ell}(F_{\overline{\omega}}\times_\omega))$ *is the functional representation space of* $\mathrm{GL}_{f_\ell\cdot\ell}(F_{\overline{\omega}}\times F_\omega)$ *;*
- $\mathrm{AdFRepsp}(\mathrm{GL}_{f_\ell\cdot\ell}(\mathbb{C}\;\times\;\mathbb{C}\,))$ *is the bilinear fibre of the tangent bibundle* $\mathrm{TAN}[\mathrm{SP}^{f_\ell}(X_R^{\rm sv})\times\mathrm{SP}^{f_\ell}(X_L^{\rm sv})]$ *introduced in definition 2.2.*

Proof. As we are concerned with mixed bimotives of the product $DM_L(k)\times DM_R(k)$ of triangulated categories, where a triangulated category is an additive category graded by a translation functor, we have that the total space $(\Delta_{T_R}^{2f_\ell\cdot\ell}\times\Delta_{T_L}^{2f_\ell\cdot\ell})$ of the tangent bibundle $\mathrm{TAN}[SP^{f_\ell}(X_{T_R}^{\rm sv}))]\times\mathrm{TAN}[SP^{f_\ell}(X_{T_L}^{\rm sv}))]$, introduced in definition 2.2, is the tangent bisemispace of $(SP^{f_\ell}(X_{T_R}^{\rm sv})\times SP^{f_\ell}(X_{T_L}^{\rm sv}))$ generated under the action of the Lie algebra of $\mathrm{GL}_{f_\ell\cdot\ell}(\mathbb{C}\,\times\mathbb{C}\,)$.
Let $\mathrm{Tan}_e(SP^{f_\ell}(X_{T_R}^{\rm sv})\times SP^{f_\ell}(X_{T_L}^{\rm sv}))$ denote this tangent bisemispace at the identity element " e " in order to define differentials on it. Then, $\mathrm{Aut}(\mathrm{Tan}_e(SP^{f_\ell}(X_{T_R}^{\rm sv})\times SP^{f_\ell}(X_{T_L}^{\rm sv})))$ is an open subset of the bilinear vector semispace of endomorphisms of $\mathrm{Tan}_e(SP^{f_\ell}(X_{T_R}^{\rm sv})\times SP^{f_\ell}(X_{T_L}^{\rm sv}))$ [F-H]. So, we have that:

$$\begin{aligned}\mathrm{Aut}(\mathrm{Tan}_e(SP^{f_\ell}(X_{T_R}^{\rm sv})\times SP^{f_\ell}(X_{T_L}^{\rm sv}))) &\subset \Delta_{T_R}^{2f_\ell\cdot\ell}\times\Delta_{T_L}^{2f_\ell\cdot\ell}\\ &\simeq \mathrm{AdFRepsp}(\mathrm{GL}_{f_\ell\cdot\ell}(\mathbb{C}\,\times\mathbb{C}\,))\times\mathrm{FRepsp}(\mathrm{GL}_{f_\ell\cdot\ell}(F_{\overline{\omega}}\times F_\omega))\\ &= \mathrm{FRepsp}(\mathrm{GL}_{f_\ell\cdot\ell}((F_{\overline{\omega}}\otimes\mathbb{C}\,)\times(F_\omega\otimes\mathbb{C}\,))))\ . \quad \blacksquare\end{aligned}$$

2.6 Weil algebra of the adjoint representation of $\mathbf{GL}_{f_\ell\cdot\ell}(\mathbb{C}\times\mathbb{C})$

Let $\mathrm{TAN}[SP^{f_\ell}(X_R^{\mathrm{sv}})\times SP^{f_\ell}(X_L^{\mathrm{sv}})]$ be the tangent bibundle having as bilinear fibre the adjoint functional representation space of $\mathrm{GL}_{f_\ell\cdot\ell}(\mathbb{C}\times\mathbb{C})$ given by:

$$\mathrm{AdFRepsp}(\mathrm{GL}_{f_\ell\cdot\ell}(\mathbb{C}\times\mathbb{C}))\simeq(\Delta_R^{2f_\ell\cdot\ell}\times\Delta_L^{2f_\ell\cdot\ell})\big/SP^{f_\ell}(X_R^{\mathrm{sv}})\times SP^{f_\ell}(X_L^{\mathrm{sv}})$$

and denoted $\mathcal{F}_{R\times L}^{2f_\ell\cdot\ell}(\mathrm{TAN})$.
The Lie algebra of $\mathcal{F}_{R\times L}^{2f_\ell\cdot\ell}(\mathrm{TAN})$ is denoted $\mathrm{Lie}(\mathcal{F}_{R\times L}^{2f_\ell\cdot\ell}(\mathrm{TAN}))$.
Let $A(\mathrm{Lie}(\mathcal{F}_{R\times L}^{2f_\ell\cdot\ell}(\mathrm{TAN})))$ be the exterior algebra of products, right by left, of differential forms of all degrees on $\mathrm{Lie}(\mathcal{F}_{R\times L}^{2f_\ell\cdot\ell}(\mathrm{TAN}))$ and let $S(\mathrm{Lie}(\mathcal{F}_{R\times L}^{2f_\ell\cdot\ell}(\mathrm{TAN})))$ denote the symmetric bialgebra corresponding to the symmetric multilinear forms on $\mathrm{Lie}(\mathcal{F}_{R\times L}^{2f_\ell\cdot\ell}(\mathrm{TAN}))$.
Then, the Weil bilinear algebra of the Lie algebra $\mathrm{Lie}(\mathcal{F}_{R\times L}^{2f_\ell\cdot\ell}(\mathrm{TAN}))$ is the graded bialgebra [G-H-V], [Hum],

$$W(\mathrm{Lie}(\mathcal{F}_{R\times L}^{2f_\ell\cdot\ell}(\mathrm{TAN})))=A(\mathrm{Lie}(\mathcal{F}_{R\times L}^{2f_\ell\cdot\ell}(\mathrm{TAN})))\times S(\mathrm{Lie}(\mathcal{F}_{R\times L}^{2f_\ell\cdot\ell}(\mathrm{TAN})))\ .$$

2.7 Definition: Connection on the tangent bisemispace

Let $\Lambda(SP^{f_\ell}(X_R^{\mathrm{sv}})\times SP^{f_\ell}(X_L^{\mathrm{sv}}))$ denote the graded differential algebra of differential forms of $SP^{f_\ell}(X_R^{\mathrm{sv}})\times SP^{f_\ell}(X_L^{\mathrm{sv}})$ and let $\Lambda(\Delta_R^{2f_\ell\cdot\ell}\times\Delta_L^{2f_\ell\cdot\ell})$ denote the graded differential algebra of differential forms of $(\Delta_R^{2f_\ell\cdot\ell}\times\Delta_L^{2f_\ell\cdot\ell})$.
A connection on the fibered tangent bisemispace $(\Delta_R^{2f_\ell\cdot\ell}\times\Delta_L^{2f_\ell\cdot\ell})$ consists in a bilinear mapping $f_{R\times L}^{\mathrm{TAN}}$ of $A^1(\mathrm{Lie}(\mathcal{F}_{R\times L}^{2f_\ell\cdot\ell}(\mathrm{TAN})))$ in the subspace of bielements of degree one of the bialgebra $\Lambda(\Delta_R^{2f_\ell\cdot\ell}\times\Delta_L^{2f_\ell\cdot\ell})$.

2.8 Proposition

Let $I_s(\mathrm{Lie}(\mathcal{F}_{R\times L}^{2f_\ell\cdot\ell}(\mathrm{TAN})))$ *denote the subalgebra of invariant elements of* $S(\mathrm{Lie}(\mathcal{F}_{R\times L}^{2f_\ell\cdot\ell}(\mathrm{TAN})))$ [Car] *which is the algebra of symmetric multilinear forms* $V(\mathrm{Lie}(\mathcal{F}_{R\times L}^{2f_\ell\cdot\ell}(\mathrm{TAN})))$ *on* $\mathrm{Lie}(\mathcal{F}_{R\times L}^{2f_\ell\cdot\ell}(\mathrm{TAN}))$.
Then, there is a homomorphism:

$$h_{R\times L}^{\mathrm{TAN}}:\quad V(\mathrm{Lie}(\mathcal{F}_{R\times L}^{2f_\ell\cdot\ell}(\mathrm{TAN})))\longrightarrow H_{2f_\ell\cdot\ell}(\Lambda(SP^{f_\ell}(X_R^{\mathrm{sv}})\times SP^{f_\ell}(X_L^{\mathrm{sv}})))\ ,$$

corresponding to the Chern-Weil homomorphism, such that a connection, associated to the homomorphism $I_s(\mathrm{Lie}(\mathcal{F}_{R\times L}^{2f_\ell\cdot\ell}(\mathrm{TAN})))\longrightarrow\Lambda(SP^{f_\ell}(X_{T_R}^{\mathrm{sv}})\times SP^{f_\ell}(X_{T_L}^{\mathrm{sv}}))$, *is equivalent to the existence of a bilinear "**contracting**" fibre* $\mathcal{F}_{R\times L}^{2f_\ell\cdot\ell}(\mathrm{TAN})$ *in the tangent bibundle* $\mathrm{TAN}[SP^{f_\ell}(X_R^{\mathrm{sv}})\times SP^{f_\ell}(X_L^{\mathrm{sv}})]$ *which implies that:*

$$H_{2f_\ell\cdot\ell,}(\Delta_{R\times L}^{2f_\ell\cdot\ell},\mathcal{F}_{R\times L}^{2f_\ell\cdot\ell}(\mathrm{TAN}))\simeq\mathrm{AdFRepsp}(\mathrm{GL}_{f_\ell\cdot\ell}(\mathbb{C}\times\mathbb{C}))$$

and thus that:

$$H^{2f_\ell\cdot\ell}[X_R^{2n_\ell}[2f_\ell\cdot\ell]\times X_L^{2n_\ell}[2f_\ell\cdot\ell],\Delta_R^{2f_\ell\cdot\ell}\times\Delta_L^{2f_\ell\cdot\ell}]\simeq \mathrm{FRepsp}(\mathrm{GL}_{f_\ell\cdot\ell}(F_{\overline{\omega}}\times\mathbb{C}\,)\otimes(F_\omega\times\mathbb{C}\,)))$$

where $X_L^{2n_\ell}[2f_\ell\cdot\ell]$ *denotes a (semi)scheme of dimension* $2n_\ell$ *shifted in* $2f_\ell\cdot\ell$ *dimensions according to* $X_L^{2n_\ell}[2f_\ell\cdot\ell]=\Delta_L^{2f_\ell\cdot\ell}\times X_L^{2n_\ell-2f_\ell\cdot\ell}$.

Proof. 1) The connection $f_{R\times L}^{\mathrm{TAN}}$ on the fibered tangent bisemispace $(\Delta_R^{2f_\ell\cdot\ell}\times\Delta_L^{2f_\ell\cdot\ell})$ can be extended to a homomorphism:

$$f_{R\times L}^{\mathrm{TAN}'}:\quad A(\mathrm{Lie}(\mathcal{F}_{R\times L}^{2f_\ell\cdot\ell}(\mathrm{TAN})))\longrightarrow\Lambda(\Delta_R^{2f_\ell\cdot\ell}\times\Delta_L^{2f_\ell\cdot\ell})\ .$$

2) According to H. Cartan [Car], the knowledge of $(SP^{f_\ell}(X_R^{\mathrm{sv}})\times SP^{f_\ell}(X_L^{\mathrm{sv}}))$ together with the connection $f_{R\times L}^{\mathrm{TAN}}$ is sufficient to know

$$H^{2f_\ell\cdot\ell}(X_R^{2n_\ell}[2f_\ell\cdot\ell]\times X_L^{2n_\ell}[2f_\ell\cdot\ell],\Delta_R^{2f_\ell\cdot\ell}\times\Delta_L^{2f_\ell\cdot\ell})\ .$$

3) Thus, the existence of a connection $f_{R\times L}^{\mathrm{TAN}}$, associated to the knowledge of $\Lambda(\Delta_R^{2f_\ell\cdot\ell}\times\Delta_L^{2f_\ell\cdot\ell})$ via the homomorphism $f_{R\times L}^{\mathrm{TAN}'}$, is equivalent to the existence of a bilinear fibre $\mathcal{F}_{R\times L}^{2f_\ell\cdot\ell}(\mathrm{TAN})$ on $(SP^{f_\ell}(X_R^{\mathrm{sv}})\times SP^{f_\ell}(X_L^{\mathrm{sv}}))$.

4) If this bilinear fibre is contracting, we have that the homology of this bilinear fibre is given by:

$$H_{2f_\ell\cdot\ell}(\Delta_R^{2f_\ell\cdot\ell}\times\Delta_L^{2f_\ell\cdot\ell},\mathcal{F}_{R\times L}^{2f_\ell\cdot\ell}(\mathrm{TAN}))\simeq\mathrm{AdFRepsp}(\mathrm{GL}_{f_\ell\cdot\ell}(\mathbb{C}\,\times\mathbb{C}\,))\ .$$

5) And thus, the bilinear cohomology with coefficients in $(\Delta_R^{2f_\ell\cdot\ell}\times\Delta_L^{2f_\ell\cdot\ell})$ must be developed according to:

$$\begin{aligned}
&H^{2f_\ell\cdot\ell}(X_R^{2n_\ell}[2f_\ell\cdot\ell]\times X_L^{2n_\ell}[2f_\ell\cdot\ell],\Delta_R^{2f_\ell\cdot\ell}\times\Delta_L^{2f_\ell\cdot\ell})\\
&\qquad=H_{2f_\ell\cdot\ell}(\Delta_R^{2f_\ell\cdot\ell}\times\Delta_L^{2f_\ell\cdot\ell},\mathcal{F}_{R\times L}^{2f_\ell\cdot\ell}(\mathrm{TAN}))\\
&\qquad\qquad\times H^{2f_\ell\cdot\ell}[X_R^{2n_\ell-2f_\ell\cdot\ell}\times X_L^{2n_\ell-2f_\ell\cdot\ell},SP^{f_\ell}(X_{T_R}^{\mathrm{sv}})\times SP(X_{T_L}^{\mathrm{sv}}))\\
&\qquad\qquad\qquad\simeq\mathrm{AdFRepsp}(\mathrm{GL}_{f_\ell\cdot\ell}(\mathbb{C}\,\times\mathbb{C}\,)\times\mathrm{FRepsp}(\mathrm{GL}_{f_\ell\cdot\ell}(F_{\overline{\omega}}\times F_\omega))
\end{aligned}$$

taking into account that [Pie3]:

$$\begin{aligned}
&H^{2f_\ell\cdot\ell}(X_R^{2n_\ell-2f_\ell\cdot\ell}\times X_L^{2n_\ell-2f_\ell\cdot\ell},SP^{f_\ell}(X_R^{\mathrm{sv}})\times SP(X_{T_L}^{\mathrm{sv}})\\
&\qquad\qquad\simeq\mathrm{FRepsp}(\mathrm{GL}_{f_\ell\cdot\ell}(F_{\overline{\omega}}\times F_\omega))\ .\quad\blacksquare
\end{aligned}$$

2.9 Definition

The bilinear Lie algebra $\mathrm{Lie}(G^{(n)}_{\mathcal{L}}(F_{\overline{\omega}}\times F_\omega))$ of the Lie bilinear semigroup $G^{(n)}_{\mathcal{L}}(F_{\overline{\omega}}\times F_\omega)$ associated with the bilinear semigroup $G^{(n)}(F_{\overline{\omega}}\times F_\omega)$ can be introduced by noting that $\mathrm{Lie}(G^{(n)}_{\mathcal{L}}(F_{\overline{\omega}}\times F_\omega))$ naturally decomposes according to:

$$\mathrm{Lie}(G^{(n)}_{\mathcal{L}}(F_{\overline{\omega}}\times F_\omega))=\mathrm{Lie}(T^{(n)}_{\mathcal{L}}(F_{\overline{\omega}}))\otimes\mathrm{Lie}(T^{(n)}_{\mathcal{L}}(F_\omega))$$

where $\mathrm{Lie}(T^{(n)}_{\mathcal{L}}(F_\omega))$ is the linear Lie algebra of the Lie semigroup $T^{(n)}_{\mathcal{L}}(F_\omega)$ associated with the linear semigroup $T^{(n)}(F_\omega)$ being the representation semispace of the group $T_n(F_\omega)$ of upper triangular matrices (see section 1.6).

The Lie algebra $\mathrm{Lie}(G^{(n)}_{\mathcal{L}}(F_{\overline{\omega}}\times F_\omega))$ corresponds to the bilinear tensor product of a vector semispace $\mathrm{Lie}(T^{(n)}_{\mathcal{L}}(F_\omega))$ by is dual $\mathrm{Lie}(T^{(n)}_{\mathcal{L}}(F_{\overline{\omega}}))$, also called a bilinear vector semispace [Pie5].

Each element of $\mathrm{Lie}(T^{(n)}_{\mathcal{L}}(F_\omega))$ (resp. $\mathrm{Lie}(T^{(n)}_{\mathcal{L}}(F_{\overline{\omega}}))$) defines a one-parameter semigroup of automorphisms of $T^{(n)}_{\mathcal{L}}(F_\omega)$ (resp. $T^{(n)}_{\mathcal{L}}(F_{\overline{\omega}})$), which are the right translations by a one-parameter subgroup of $T^{(n)}_{\mathcal{L}}(F_\omega)$ (resp. $T^{(n)}_{\mathcal{L}}(F_{\overline{\omega}})$).

More exactly, the bilinear Lie algebra $\mathrm{Lie}(G^{(n)}_{\mathcal{L}}(F_{\overline{\omega}}\times F_\omega))$ is defined by the two conditions:

a) $\mathrm{Lie}(G^{(n)}_{\mathcal{L}}(F_{\overline{\omega}}\times F_\omega))$ is a bilinear vector semispace over the product $F_{\overline{\omega}}\times F_\omega$ of sets of completions;

b) to each pair $(\tau^t_{F_{\overline{\omega}}},\tau_{F_\omega})$, with $\tau^t_{F_{\overline{\omega}}}\in\mathrm{Lie}(T^{(n)}_{\mathcal{L}}(F_{\overline{\omega}}))$ and $\tau_{F_\omega}\in\mathrm{Lie}(T^{(n)}_{\mathcal{L}}(F_\omega))$, corresponds an element of $\mathrm{Lie}(G^{(n)}_{\mathcal{L}}(F_{\overline{\omega}}\times F_\omega))$, noted $[\tau^t_{F_{\overline{\omega}}},\tau_{F_\omega}]$

- which is linear with respect to $\tau^t_{F_{\overline{\omega}}}$ and to τ_{F_ω};
- whose value is given by $[\tau^t_{F_{\overline{\omega}}},\tau_{F_\omega}]=\tau^t_{F_{\overline{\omega}}}\centerdot\tau_{F_\omega}-\tau_{F_\omega}\centerdot\tau^t_{F_{\overline{\omega}}}$;
- which verifies the Jacobi identity.

2.10 Proposition

The bilinear cohomology with coefficients in the tangent bisemispace $(\Delta_R^{2f_\ell\cdot\ell}\times\Delta_L^{2f_\ell\cdot\ell})$, noted $H^{2f_\ell\cdot\ell}(X_R^{2n_\ell}[2f_\ell\cdot\ell]\times X_L^{2n_\ell}[2f_\ell\cdot\ell],\Delta_R^{2f_\ell\cdot\ell}\times\Delta_L^{2f_\ell\cdot\ell})$, is in one-to-one correspondence with the Lie algebra of the general bilinear semigroup $\mathrm{GL}_{f_\ell\cdot\ell}(F_{\overline{\omega}}\times F_\omega)$:

$$H^{2f_\ell\cdot\ell}(X_R^{2n_\ell}[2f_\ell\cdot\ell]\times X_L^{2n_\ell}[2f_\ell\cdot\ell],\Delta_R^{2f_\ell\cdot\ell}\times\Delta_L^{2f_\ell\cdot\ell})\simeq\mathrm{Lie}(\mathrm{GL}_{f_\ell\cdot\ell}(F_{\overline{\omega}}\times F_\omega))\;.$$

Proof. Indeed, according to proposition 2.8, we have that

$$\begin{aligned}
H^{2f_\ell\cdot\ell}(X_R^{2n_\ell}[2f_\ell\cdot\ell]\times X_L^{2n_\ell}[2f_\ell\cdot\ell],\Delta_R^{2f_\ell\cdot\ell}\times\Delta_L^{2f_\ell\cdot\ell})&\\
&\simeq \mathrm{FRepsp}(\mathrm{GL}_{f_\ell\cdot\ell}(F_{\overline{\omega}}\otimes\mathbb{C}\,)\times(F_\omega\otimes\mathbb{C}\,)))\\
&= \mathrm{AdFRepsp}(\mathrm{GL}_{f_\ell\cdot\ell}(\mathbb{C}\,\times\mathbb{C}\,))\times\mathrm{FRepsp}(\mathrm{GL}_{f_\ell\cdot\ell}(F_{\overline{\omega}}\times F_\omega))
\end{aligned}$$

from which it clearly appears that:

$$\mathrm{Lie}(\mathrm{GL}_{f_\ell\cdot\ell}(F_{\overline{\omega}}\times F_\omega)=\mathrm{FRepsp}(\mathrm{GL}_{f_\ell\cdot\ell}(F_{\overline{\omega}}\otimes\mathbb{C}\,)\times(F_\omega\otimes\mathbb{C}\,))$$

since the bilinear fibre $\mathcal{F}_{R\times L}^{2f_\ell\cdot\ell}(\mathrm{TAN})$ of the tangent bibundle $\mathrm{TAN}[SP^{f_\ell}(X_R^{\mathrm{sv}})\times SP^{f_\ell}(X_L^{\mathrm{sv}})]$ is precisely the adjoint functional representation space $\mathrm{AdFRepsp}(\mathrm{GL}_{f_\ell\cdot\ell}(\mathbb{C}\,\times\mathbb{C}\,))$ of $\mathrm{GL}_{f_\ell\cdot\ell}(\mathbb{C}\,\times\mathbb{C}\,)$. ■

2.11 Proposition

Let $H^{2n_\ell}(\underline{c}_*(X_{R\times L}^{\mathrm{sv}}),Z_{R\times L}(2n_\ell))$ be the bilinear cohomology of the Suslin-Voevodsky pure bisemimotives $\underline{c}_*(X_{T_{R\times L}}^{\mathrm{sv}})$ with coefficients in the product, right by left, of Suslin-Voevodsky sub(bisemi)motives of complex codimension n_ℓ.

Then, the cohomology of the corresponding Suslin-Voevodsky mixed bisemimotive $M_{DM_R}(X_R^{\mathrm{sv}})\times M_{DM_L}(X_L^{\mathrm{sv}})$, noted $M_{DM_{R\times L}}(X_{R\times L}^{\mathrm{sv}})$, can be reached throughout the following endomorphism:

$$\begin{aligned}
\mathbb{H}\,D_{2f_\ell\cdot\ell}:\quad & H^{2n_\ell}(\underline{c}_*(X_{R\times L}^{\mathrm{sv}}),Z_{R\times L}(2n_\ell))\\
&\longrightarrow H^{2n_\ell-2f_\ell\cdot\ell}(M_{DM_{R\times L}}(X_{R\times L}^{\mathrm{sv}}),Z_{R\times L}(2n_\ell[2f_\ell\cdot\ell])
\end{aligned}$$

where $Z_{R\times L}(2n_\ell[2f_\ell\cdot\ell])$ is the product, right by left, of Suslin-Voevodsky mixed sub-bisemimotives of complex codimension n_ℓ shifted in $f_\ell\cdot\ell$ complex dimensions and written $(X_R^{2n_\ell}[2f_\ell\cdot\ell]\times X_L^{2n_\ell}[2f_\ell\cdot\ell])$ in section 2.4, such that the cohomology of the Suslin-Voevodsky mixed bisemimotive decomposes according to:

$$\begin{aligned}
H^{2n_\ell-2f_\ell\cdot\ell}&(M_{DM_{R\times L}}(X_{R\times L}^{\mathrm{sv}}),Z_{R\times L}(2n_\ell[2f_\ell\cdot\ell])\\
&=H_{2f_\ell\cdot\ell}(\Delta_R^{2f_\ell\cdot\ell}\times\Delta_L^{2f_\ell\cdot\ell},\mathcal{F}_{R\times L}^{2f_\ell\cdot\ell}(\mathrm{TAN}))\times H^{2n_\ell}(\underline{c}_*(X_{R\times L}^{\mathrm{sv}}),Z_{R\times L}(2n_\ell))\,.
\end{aligned}$$

Proof. Taking into account that $X_L^{2n_\ell}[2f_\ell\cdot\ell]=\Delta_L^{2f_\ell\cdot\ell}\times X_L^{2n_\ell-2f_\ell\cdot\ell}$ (resp. $X_R^{2n_\ell}[2f_\ell\cdot\ell]=\Delta_R^{2f_\ell\cdot\ell}\times X_R^{2n_\ell-2f_\ell\cdot\ell}$) according to section 2.4, we have that the cohomology of $\underline{c}_*(X_{R\times L}^{\mathrm{sv}})$,

submitted to a translation functor acted by the tangent bibundle $\mathrm{TAN}[SP^{f_\ell}(X_R^{\rm sv})\times SP^{f_\ell}(X_L^{\rm sv})]$, is transformed according to:

$$\begin{aligned}
\mathbb{H}\,D_{2f_\ell\cdot\ell}:\quad & H^{2n_\ell}(\underline{c}_*(X^{\rm sv}_{R\times L}),Z_{R\times L}(2n_\ell))\\
&\longrightarrow H^{2f_\ell\cdot\ell}(M_{DM_{R\times L}}(X^{\rm sv}_{R\times L}),\Delta_R^{2f_\ell\cdot\ell}\times\Delta_L^{2f_\ell\cdot\ell})\\
&\qquad\oplus H^{2n_\ell-2f_\ell\cdot\ell}(\underline{c}_*(X^{\rm sv}_{R\times L}),X_R^{2n_\ell-2f_\ell\cdot\ell}\times X_L^{2n_\ell-2f_\ell\cdot\ell})\\
&= (H_{2f_\ell\cdot\ell}(\Delta_R^{2f_\ell\cdot\ell}\times\Delta_L^{2f_\ell\cdot\ell},\mathcal{F}^{2f_\ell\cdot\ell}_{R\times L}(\mathrm{TAN})))\\
&\qquad\times\big[H^{2f_\ell\cdot\ell}(\underline{c}_*(X^{\rm sv}_{R\times L}),SP^{f_\ell}(X_R^{\rm sv})\times SP^{f_\ell}(X_L^{\rm sv}))\\
&\qquad\oplus H^{2n_\ell-2f_\ell\cdot\ell}(\underline{c}_*(X^{\rm sv}_{R\times L}),X_R^{2n_\ell-2f_\ell\cdot\ell}\times X_L^{2n_\ell-2f_\ell\cdot\ell})\big]\\
&= H_{2f_\ell\cdot\ell}(\Delta_R^{2f_\ell\cdot\ell}\times\Delta_L^{2f_\ell\cdot\ell},\mathcal{F}^{2f_\ell\cdot\ell}_{R\times L}(\mathrm{TAN}))\\
&\qquad\times H^{2n_\ell}(\underline{c}_*(X^{\rm sv}_{R\times L}),Z_{R\times L}(2n_\ell))
\end{aligned}$$

such that $H_{2f_\ell\cdot\ell}(\Delta_R^{2f_\ell\cdot\ell}\times\Delta_R^{2f_\ell\cdot\ell},\mathcal{F}^{2f_\ell\cdot\ell}_{R\times L}(\mathrm{TAN}))\times H^{2n_\ell}(\underline{c}_*(X^{\rm sv}_{R\times L}),Z_{R\times L}(2n_\ell))$ be noted $H^{2n_\ell-2f_\ell\cdot\ell}(M_{DM_{R\times L}}(X^{\rm sv}_{R\times L}),Z_{R\times L}(2n_\ell[2f_\ell\cdot\ell])$.
These equalities essentially result from proposition 2.8.
It then results that the cohomology $H^{2n_\ell-2f_\ell\cdot\ell}(M_{DM_{R\times L}}(X^{\rm sv}_{R\times L}),Z_{R\times L}(2n_\ell[2f_\ell\cdot\ell])$ of the Suslin-Voevodsky mixed bisemimotive decomposes into the $(2f_\ell\cdot\ell)$-homology with coefficients in the bilinear fibre $\mathcal{F}^{f_\ell\cdot\ell}_{R\times L}(\mathrm{TAN})$ acting on the $(2n_\ell)$-cohomology of the Suslin-Voevodsky pure bisemimotive $\underline{c}_*(X^{\rm sv}_{R\times L})$. ■

2.12 Proposition

The bilinear cohomology of the Suslin-Voevodsky mixed bisemimotive $M_{DM_{R\times L}}(X^{\rm sv}_{R\times L})$ is in bijection with the functional representation space of the bilinear general semigroup $\mathrm{GL}_{n_\ell}(F_{\overline{\omega}}\times F_\omega)$ shifted in $f_\ell\cdot\ell$-complex dimensions:

$$\begin{aligned}
&H^{2n_\ell-2f_\ell\cdot\ell}(M_{DM_{R\times L}}(X^{\rm sv}_{R\times L}),Z_{R\times L}(2n_\ell[2f_\ell\cdot\ell]])\\
&\qquad\simeq \mathrm{FRepsp}(\mathrm{GL}_{n_\ell[f_\ell\cdot\ell]}((F_{\overline{\omega}}\otimes\mathbb{C}\,)\times((F_\omega\otimes\mathbb{C}\,))
\end{aligned}$$

where $\mathrm{FRepsp}(\mathrm{GL}_{n_\ell[f_\ell\cdot\ell]}((F_{\overline{\omega}}\otimes\mathbb{C}\,)\times((F_\omega\otimes\mathbb{C}\,)$ *is a condensed notation for* $\mathrm{AdFRepsp}(\mathrm{GL}_{f_\ell\cdot\ell}(\mathbb{C}\times\mathbb{C}\,)\times\mathrm{FRepsp}(\mathrm{GL}_{n_\ell}(F_{\overline{\omega}}\times F_\omega))$.

Proof. According to proposition 2.11, we have that:

$$\begin{aligned}
&H^{2n_\ell-2f_\ell\cdot\ell}(M_{DM_{R\times L}}(X^{\rm sv}_{R\times L}),Z_{R\times L}(2n_\ell[2f_\ell\cdot\ell])\\
&\qquad= (H_{2f_\ell\cdot\ell}(\Delta_R^{2f_\ell\cdot\ell}\times\Delta_L^{2f_\ell\cdot\ell},\mathcal{F}^{2f_\ell\cdot\ell}_{R\times L}(\mathrm{TAN}))\\
&\qquad\quad\times\big[H^{2f_\ell\cdot\ell}(\underline{c}_*(X^{\rm sv}_{R\times L}),SP^{f_\ell}(X_R^{\rm sv})\times SP^{f_\ell}(X_L^{\rm sv}))\\
&\qquad\quad\oplus H^{2n_\ell-2f_\ell\cdot\ell}(\underline{c}_*(X^{\rm sv}_{R\times L}),X_R^{2n_\ell-2f_\ell\cdot\ell}\times X_L^{2n_\ell-2f_\ell\cdot\ell}))\big]\,.
\end{aligned}$$

And, propositions 2.8 and 2.10 give the following isomorphisms:

- $H_{2f_\ell\cdot\ell}(\Delta_R^{2f_\ell\cdot\ell}\times\Delta_L^{2f_\ell\cdot\ell},\mathcal{F}_{R\times L}^{2f_\ell\cdot\ell}(\mathrm{TAN}))\simeq\mathrm{AdFRepsp}(\mathrm{GL}_{f_\ell\cdot\ell}(\mathbb{C}\times\mathbb{C}))$;
- $H^{2f_\ell\cdot\ell}(\underline{c}_*(X^{\mathrm{sv}}_{R\times L}),SP^{f_\ell}(X^{\mathrm{sv}}_R)\times SP^{f_\ell}(X^{\mathrm{sv}}_L))\simeq\mathrm{FRepsp}(\mathrm{GL}_{f_\ell\cdot\ell}(F_{\overline{\omega}}\times F_\omega))$;
- $H^{2n_\ell-2f_\ell\cdot\ell}(\underline{c}_*(X^{\mathrm{sv}}_{R\times L}),X_R^{2n_\ell-2f_\ell\cdot\ell}\times X_L^{2n_\ell-2f_\ell\cdot\ell})\simeq\mathrm{FRepsp}(\mathrm{GL}_{n_\ell-f_\ell\cdot\ell}(F_{\overline{\omega}}\times F_\omega))$;

leading to:

$$\begin{aligned}
&H^{2n_\ell-2f_\ell\cdot\ell}(M_{DM_{R\times L}}(X^{\mathrm{sv}}_{R\times L}),Z_{R\times L}(2n_\ell[2f_\ell\cdot\ell])\\
&\qquad\simeq\mathrm{AdFRepsp}(\mathrm{GL}_{f_\ell\cdot\ell}(\mathbb{C}\times\mathbb{C}))\\
&\qquad\qquad\times\left[\mathrm{FRepsp}(\mathrm{GL}_{f_\ell\cdot\ell}(F_{\overline{\omega}}\times F_\omega))\oplus\mathrm{FRepsp}(\mathrm{GL}_{n_\ell-f_\ell\cdot\ell}(F_{\overline{\omega}}\times F_\omega))\right]\\
&\qquad=\mathrm{AdFRepsp}(\mathrm{GL}_{f_\ell\cdot\ell}(\mathbb{C}\times\mathbb{C}))\times\mathrm{FRepsp}(\mathrm{GL}_{n_\ell}(F_{\overline{\omega}}\times F_\omega))
\end{aligned}$$ ■

2.13 Higher Chow semigroups [Blo], [Gil1]

- According to proposition 1.17, the left (resp. right) Suslin-Voevodsky subsemimotive $Z_L(2n_\ell)$ (resp. $Z_R(2n_\ell)$) of complex dimension n_ℓ can be isomorphic to the left (resp. right) semicycle $\mathrm{CY}^{2n_\ell}(Y_L)$ (resp. $\mathrm{CY}^{2n_\ell}(Y_R)$) belonging to the $2n_\ell$-th Chow semigroup $\mathrm{CH}^{2n_\ell}(Y_L)$ (resp. $\mathrm{CH}^{2n_\ell}(Y_R)$):

$$Z_L(2n_\ell)\simeq\mathrm{CY}^{2n_\ell}(Y_L)\in\mathrm{CH}^{2n_\ell}(Y_L)$$

$$(\text{resp.}\quad Z_R(2n_\ell)\simeq\mathrm{CY}^{2n_\ell}(Y_R)\in\mathrm{CH}^{2n_\ell}(Y_R)\ )\ .$$

- Similarly, the left (resp. right) Suslin-Voevodsky mixed submotive $Z_L(2n_\ell[2f_\ell\cdot\ell])$ (resp. $Z_R(2n_\ell[2f_\ell\cdot\ell])$) of complex dimension n_ℓ, shifted in $f_\ell\cdot\ell$ complex dimensions, can be isomorphic to the left (resp. right) cycle $\mathrm{CY}^{2n_\ell}(Y_L,[2f_\ell\cdot\ell])$ (resp. $\mathrm{CY}^{2n_\ell}(Y_L,[2f_\ell\cdot\ell])$) of complex dimension n_ℓ, shifted in $f_\ell\cdot\ell$ complex dimensions, belonging to the $2n_\ell$-th higher Chow semigroup $\mathrm{CH}^{2n_\ell}(Y_L,[2f_\ell\cdot\ell])$ (resp. $\mathrm{CH}^{2n_\ell}(Y_R,[2f_\ell\cdot\ell])$):

$$Z_L(2n_\ell[2f_\ell\cdot\ell])\simeq\mathrm{CY}^{2n_\ell}(Y_L,[2f_\ell\cdot\ell])\in\mathrm{CH}^{2n_\ell}(Y_L,[2f_\ell\cdot\ell])$$

$$(\text{resp.}\quad Z_R(2n_\ell[2f_\ell\cdot\ell])\simeq\mathrm{CY}^{2n_\ell}(Y_R,[2f_\ell\cdot\ell])\in\mathrm{CH}^{2n_\ell}(Y_R,[2f_\ell\cdot\ell])\ ).$$

2.14 Representation of the general bilinear shifted semigroup

According to section 1.15, the product, right by left, of cycles $\mathrm{CY}^{2n_\ell}(Y_R)\times \mathrm{CY}^{2n_\ell}(Y_L)$ leads to:

$$Z_R(2n_\ell)\times Z_L(2n_\ell)\simeq \mathrm{CY}^{2n_\ell}(Y_R)\times \mathrm{CY}^{2n_\ell}(Y_L)$$

$$\simeq \mathrm{FRepsp}(\mathrm{GL}_{n_\ell}(F_{\overline{\omega}}\times F_\omega))\ .$$

By the same way, the product, right by left, of the Suslin-Voevodsky mixed subsemimotives of complex dimension n_ℓ shifted in $f_\ell\cdot\ell$ complex dimensions gives rise to the bijections:

$$\begin{aligned}Z_{R\times L}(2n_\ell[2f_\ell\cdot\ell]) &\simeq \mathrm{CY}^{2n_\ell}(Y_R[2f_\ell\cdot\ell])\times \mathrm{CY}^{2n_\ell}(Y_L[2f_\ell\cdot\ell])\\ &\simeq \mathrm{FRepsp}(\mathrm{GL}_{n_\ell[f_\ell\cdot\ell]}((F_{\overline{\omega}}\otimes\mathbb{C}\,)\times(F_\omega\otimes\mathbb{C}\,))\ .\end{aligned}$$

2.15 Proposition

Taking into account the isomorphism i_{M-X} between the bilinear cohomology of the Suslin-Voevodsky pure bisemimotive $\underline{c}_*(X^{\mathrm{sv}}_{R\times L})$ and the bilinear cohomology of $Y_{R\times L}$ introduced in proposition 1.17, as well as the endomorphism $\mathbb{H}\,D_{2f_\ell\cdot\ell}$ between the bilinear cohomology of $\underline{c}_*(X^{\mathrm{sv}}_{R\times L})$ and the corresponding cohomology of the Suslin-Voevodsky mixed bisemimotive $M_{DM_{R\times L}}(X^{\mathrm{sv}}_{R\times L})$, we are led to the following commutative diagram:

$$\begin{array}{ccc}
H^{2n_\ell}(\underline{c}_*(X^{\mathrm{sv}}_{R\times L}),Z_{R\times L}(2n_\ell)) & \xrightarrow{i_{M-X}} & H^{2n_\ell}(Y_{R\times L},\mathrm{CY}^{2n_\ell}(Y_R)\times \mathrm{CY}^{2n_\ell}(Y_L))\\
{\scriptstyle \mathbb{H}\,D_{2f_\ell\cdot\ell}}\big\downarrow & & \big\downarrow{\scriptstyle \mathbb{H}\,D^{X-M}_{2f_\ell\cdot\ell}}\\
H^{2n_\ell-2f_\ell\cdot\ell}(M_{DM_{R\times L}}(X^{\mathrm{sv}}_{R\times L}), & \xrightarrow{i^{\mathrm{sc}}_{M-X}} & H^{2n_\ell-2f_\ell\cdot\ell}(Y_{R\times L}[2f_\ell\cdot\ell],\\
Z_{R\times L}(2n_\ell[2f_\ell\cdot\ell]) & & \mathrm{CY}^{2n_\ell}(Y_R[2f_\ell\cdot\ell])\times \mathrm{CY}^{2n_\ell}(Y_L[2f_\ell\cdot\ell])
\end{array}$$

where $Y_{R\times L}[2f_\ell\cdot\ell]$ is the bisemigroup $Y_{R\times L}$ shifted in $f_\ell\cdot\ell$ complex dimensions on its right and left parts.

2.16 Bilinear mixed cohomology

The introduction in this chapter of the bilinear cohomology of mixed bisemimotives naturally leads to precise what must be a general bilinear mixed (or shifted) cohomology referring to the introduction of a general bilinear cohomology in section 3.2 of [Pie3] and taking into account the isomorphisms:

$$\begin{aligned}Z_{R\times L}(2n_\ell[2f_\ell\cdot\ell]) &\simeq \mathrm{CY}^{2n_\ell}(Y_{R\times L}[2f_\ell\cdot\ell])\\ &\simeq \mathrm{FRepsp}(\mathrm{GL}_{n_\ell[f_\ell\cdot\ell]}(F_{\overline{\omega}}\otimes\mathbb{C}\,)\times(F_\omega\otimes\mathbb{C}\,))\end{aligned}$$

and the bilinear mixed homology

$$H_{2f_\ell\cdot\ell}(\Delta^{2f_\ell\cdot\ell}_{R\times L},\mathcal{F}^{2f_\ell\cdot\ell}_{R\times L}(\mathrm{TAN}))\simeq \mathrm{AdFRepsp}(\mathrm{GL}_{f_\ell\cdot\ell}(\mathbb{C}\times\mathbb{C}))$$

associated with the tangent bibundle $\mathrm{TAN}[\mathrm{SP}^{f_\ell}(X^{\mathrm{sv}}_{R\times L})]$.

2.17 Proposition

A general bilinear mixed cohomology theory is a contravariant bifunctor*:*

$$\begin{aligned}\mathbb{H}^{2i-2k}:\quad & \{\textit{smooth abstract shifted bisemivarieties } G^{(n)}((F^+_{\overline{v}}\otimes\mathbb{R})\times(F^+_v\otimes\mathbb{R}))\}\\ &\longrightarrow \{\textit{graded functional representation spaces of the complete shifted}\\ &\qquad \textit{bilinear semigroups } \mathrm{GL}_{2i[2k]}((F^+_{\overline{v}}\otimes\mathbb{R})\times(F^+_v\otimes\mathbb{R}))\}\ , 0\le k\le i\ ,\end{aligned}$$

given by

$$H^{2i-2k}(G^{(n)}((F^+_{\overline{v}}\otimes\mathbb{R})\times(F^+_v\otimes\mathbb{R})),\mathrm{FRepsp}(\mathrm{GL}_{2i[2k]}((F^+_{\overline{v}}\otimes\mathbb{R})\times(F^+_v\otimes\mathbb{R})))$$

where

$$\begin{aligned}&\mathrm{FRepsp}(\mathrm{GL}_{2i[2k]}((F^+_{\overline{v}}\otimes\mathbb{R})\times(F^+_v\otimes\mathbb{R})))\\ &\qquad\qquad = \mathrm{AdFRepsp}(\mathrm{GL}_{2k}(\mathbb{R}\times\mathbb{R}))\times\mathrm{Repsp}(\mathrm{GL}_{2i}(F^+_{\overline{v}}\times F^+_v))\ .\end{aligned}$$

This general bilinear mixed cohomology is characterized by:

a) **isomorphic embeddings**

$$\begin{aligned}&G^{(n)}((F^+_{\overline{v}}\otimes\mathbb{R})\times(F^+_v\otimes\mathbb{R}))\\ &\qquad \overset{\sim}{\hookrightarrow}\quad G^{(n)}((F_{\overline{\omega}}\otimes\mathbb{C})\times(F_\omega\otimes\mathbb{C}))\\ &\mathrm{FRepsp}(\mathrm{GL}_{2i[2k]}((F^+_{\overline{v}}\otimes\mathbb{R})\times(F^+_v\otimes\mathbb{R})))\\ &\qquad \overset{\sim}{\hookrightarrow}\quad \mathrm{FRepsp}(\mathrm{GL}_{2i[2k]}((F_{\overline{\omega}}\otimes\mathbb{C})\times(F_\omega\otimes\mathbb{C})))\end{aligned}$$

of "real" shifted bisemivarieties into their complex equivalents.

b) **mixed (or shifted) bisemicycle maps***:*

$$\begin{aligned}\gamma^{i[k]}_{G^{(n)}_{\overline{v}\times v}}:\quad & \not{Z}^{i[k]}(G^{(n)}((F^+_{\overline{v}}\otimes\mathbb{R})\times(F^+_v\otimes\mathbb{R})))\\ \longrightarrow\ & H^{2i-2k}(G^{(n)}((F^+_{\overline{v}}\otimes\mathbb{R})\times(F^+_v\otimes\mathbb{R})),\mathrm{FREPSP}(\mathrm{GL}_{2i[2k]}((F^+_{\overline{v}}\otimes\mathbb{R})\times(F^+_v\otimes\mathbb{R}))))\end{aligned}$$

where $\not{Z}^{i[k]}$ *denotes the bilinear semigroup of mixed bisemicycles of codimension* i *shifted in* k *dimensions.*

c) **Hodge mixed (or shifted) bisemicycles***:*

$$H^{2i-2k}(G^{(n)}((F_{\overline{\omega}}\otimes\mathbb{C}\,)\times(F_\omega\otimes\mathbb{C}\,)),\mathrm{FREPSP}(\mathrm{GL}_{2i[2k]}(F^+_{\overline{v}}\otimes\mathbb{R}\,)\times(F^+_v\otimes\mathbb{R}\,)))$$

from the abstract "complex" shifted bisemivariety $G^{(n)}((F_{\overline{\omega}}\otimes\mathbb{C}\,)\times(F_\omega\otimes\mathbb{C}\,))$ *to the functional representation space of the "real"" shifted general bilinear semigroup* $\mathrm{GL}_{2i[2k]}(F^+_{\overline{v}}\otimes\mathbb{R}\,)\times(F^+_v\otimes\mathbb{R}\,))$ *.*

There is the shifted bifiltration $F^{p[r]}_{R\times L}$ *given by:*

$$\begin{aligned}F^{p[r]}_{R\times L}:\quad & H^{2i-2k}(G^{(n)}(\bullet\times\bullet),-)\\ &= \bigoplus_{\substack{i=p+q\\ k=r-s}} H^{(2p-2r)+(2q-2s)}(G^{(n)}((F_{\overline{\omega}}\otimes\mathbb{C}\,)\times(F_\omega\otimes\mathbb{C}\,)),\\ &\qquad\qquad \mathrm{FREPSP}(\mathrm{GL}_{2p[2r]+2q[2s]}(F^+_{\overline{v}}\otimes\mathbb{R}\,)\times(F^+_v\otimes\mathbb{R}\,)))\ .\end{aligned}$$

d) **a Künneth standard conjecture***:*

implying that the projectors on $H^{2i-2k}(G^{(n)}(\bullet\ \times\ \bullet),-)$ *(see* [Pie3]*) are induced by mixed (or shifted) compactified bisemicycles* $\mathrm{CY}^{i[k]}(G^{(n)}(F^+_{\overline{v}}\otimes\mathbb{R}\,)\times(F^+_v\otimes\mathbb{R}\,))\subset \not{Z}^{i[k]}(G^{(n)}(\bullet\ \times\ \bullet))$ *decomposing into rational mixed (or shifted) subbisemicycles according to the conjucacy class representatives of* $\mathrm{GL}_{2i[2k]}((F^+_{\overline{v}}\otimes\mathbb{R}\,)\times(F^+_v\otimes\mathbb{R}\,))$ *.*

e) **a Künneth biisomorphism***:*

$$\begin{aligned}& H^{2i-2k}(G^{(n)}(F^+_{\overline{v}}\otimes\mathbb{R}\,),\mathrm{FREPSP}(\mathrm{GL}_{2i[2k]}(F^+_{\overline{v}}\otimes\mathbb{R}\,)))\\ &\qquad \otimes_{F^+_{\overline{v}}\times F^+_v} H^{2i-2k}(G^{(n)}(F^+_v\otimes\mathbb{R}\,),\mathrm{FREPSP}(\mathrm{GL}_{2i[2k]}(F^+_v\otimes\mathbb{R}\,)))\\ &\qquad\qquad \longrightarrow H^{2i-2k}(G^{(n)}(F^+_{\overline{v}}\otimes\mathbb{R}\,)\times(F^+_v\otimes\mathbb{R}\,),\\ &\qquad\qquad\qquad \mathrm{FREPSP}(\mathrm{GL}_{2i[2k]}(F^+_{\overline{v}}\otimes\mathbb{R}\,)))\times(F^+_v\otimes\mathbb{R}\,)))\end{aligned}$$

in such a way that

$$\begin{aligned}& H^{2p-2r}(G^{(n)}(F^+_{\overline{v}}\otimes\mathbb{R}\,)\times(F^+_v\otimes\mathbb{R}\,),\mathrm{FREPSP}(\mathrm{GL}_{2p[2r]}(F^+_{\overline{v}}\otimes\mathbb{R}\,)\times(F^+_v\otimes\mathbb{R}\,)))\\ &\qquad \otimes H^{(2i-2k)-(2p-2r)}(G^{(n)}(F^+_{\overline{v}}\otimes\mathbb{R}\,)\times(F^+_v\otimes\mathbb{R}\,),\\ &\qquad\quad \mathrm{FREPSP}(\mathrm{GL}_{(2i[2k]-(2p[2r])}(F^+_{\overline{v}}\otimes\mathbb{R}\,)))\times(F^+_v\otimes\mathbb{R}\,))\\ &\qquad\qquad \longrightarrow H^0(G^{(n)}(F^+_{\overline{v}}\otimes\mathbb{R}\,)\times(F^+_v\otimes\mathbb{R}\,),\\ &\qquad\qquad\qquad \mathrm{FREPSP}(\mathrm{GL}_1(F^+_{\overline{v}}\otimes\mathbb{R}\,)))\times(F^+_v\times\mathbb{R}\,))\end{aligned}$$

is the bilinear version of the mixed intersection cohomology according to section 3.2 of [Pie3]*.*

Proof. The introduction of the general bilinear mixed cohomology follows from the introduction of general bilinear cohomology in section 3.2 of [Pie3] to which we refer. ∎

3 Bilinear K-homology associated with an elliptic bi-operator

3.1 Modular conjugacy classes of Suslin-Voevodsky pure sub-semimotives

Let $Z_L(2n_\ell) \equiv X_L^{2n_\ell}$ (resp. $Z_R(2n_\ell) \equiv X_R^{2n_\ell}$) be a Suslin-Voevodsky left (resp. right) pure subsemimotive of complex dimension n_ℓ, i.e. a left (resp. right) subpresheaf with transfers (or correspondences).
Referring to sections 2.1 and 2.14, we have that:

$$\begin{aligned} X_R^{2n_\ell} \times X_L^{2n_\ell} &\simeq \mathrm{FRepsp}(\mathrm{GL}_{n_\ell}(F_{\overline{\omega}} \times F_\omega)) \\ &\simeq \mathrm{CY}^{2n_\ell}(Y_R) \times \mathrm{CY}^{2n_\ell}(Y_L) \\ &= \{\mathrm{CY}^{2n_\ell}(Y_R(j,m_j)) \times \mathrm{CY}^{2n_\ell}(Y_L(j,m_j))\}_{j,m_j} \end{aligned}$$

in such a way that the product, right by left, of $2n_\ell$-dimensional semicycles decomposes according to the set of conjugacy class representatives of $\mathrm{GL}_{n_\ell}(F_{\overline{\omega}} \times F_\omega)$.
It follows that the product, right by left, $X_R^{2n_\ell} \times X_L^{2n_\ell}$ of Suslin-Voevodsky subsemimotives of complex dimension n_ℓ also decomposes according to the conjugacy class representatives of $\mathrm{GL}_{n_\ell}(F_{\overline{\omega}} \times F_\omega)$:

$$X_R^{2n_\ell} \times X_L^{2n_\ell} = \{X_R^{2n_\ell}(j,m_j) \times X_L^{2n_\ell}(j,m_j)\}_{j,m_j} \ .$$

3.2 Modular conjugacy classes of Suslin-Voevodsky mixed sub-semimotives

Let

$$Z_L(2n_\ell[2f_\ell \cdot \ell]) \equiv X_L^{2n_\ell}[2f_\ell \cdot \ell] \ , \qquad (\text{resp.} \quad Z_R(2n_\ell[2f_\ell \cdot \ell]) \equiv X_R^{2n_\ell}[2f_\ell \cdot \ell] \ )$$

be the left (resp. right) Suslin-Voevodsky mixed subsemimotive of complex dimension n_ℓ shifted in $2f_\ell \cdot \ell$ dimensions.
Then, as in section 3.1, we have that:

$$\begin{aligned} X_R^{2n_\ell}[2f_\ell \cdot \ell] \times X_L^{2n_\ell}[2f_\ell \cdot \ell] &\simeq \mathrm{FRepsp}(\mathrm{GL}_{n_\ell[f_\ell \cdot \ell]}(F_{\overline{\omega}} \otimes \mathbb{C}\,) \times (F_\omega \otimes \mathbb{C}\,)) \\ &\simeq \mathrm{CY}^{2n_\ell}(Y_R[2f_\ell \cdot \ell]) \times \mathrm{CY}^{2n_\ell}(Y_L[2f_\ell \cdot \ell]) \\ &= \{\mathrm{CY}^{2n_\ell}(Y_R,[2f_\ell \cdot \ell],(j,m_j)) \times \mathrm{CY}^{2n_\ell}(Y_L,[2f_\ell \cdot \ell],(j,m_j))\}_{j,m_j} \end{aligned}$$

where $\mathrm{CY}^{2n_\ell}(Y_R,[2f_\ell \cdot \ell],(j,m_j))$ is the m_j-th representative of the j-th conjugacy class of the $2n_\ell$-th (semi)cycle CY^{2n_ℓ} shifted in $2f_\ell \cdot \ell$ dimensions of the semigroup Y_R.

3.3 Definition: Differential bioperator $D_R^{2f_\ell\cdot\ell}\otimes D_L^{2f_\ell\cdot\ell}$

Let $D_R^{2f_\ell\cdot\ell}\otimes D_L^{2f_\ell\cdot\ell}$ be the product of a right linear differential (elliptic) operator $D_R^{2f_\ell\cdot\ell}$ acting on $2f_\ell\cdot\ell$ variables by its left equivalent. This bioperator is defined by its biaction

$$D_R^{2f_\ell\cdot\ell}\otimes D_L^{2f_\ell\cdot\ell}: \qquad X_R^{2n_\ell}\times X_L^{2n_\ell} \longrightarrow X_R^{2n_\ell}[2f_\ell\cdot\ell]\times X_L^{2n_\ell}[2f_\ell\cdot\ell]$$

from the Suslin-Voevodsky pure subbisemimotive $X_R^{2n_\ell}\times X_L^{2n_\ell}$ to the corresponding mixed subbisemimotive $X_R^{2n_\ell}[2f_\ell\cdot\ell]\times X_L^{2n_\ell}[2f_\ell\cdot\ell]$ shifted in $(f_\ell\cdot\ell)$ complex dimensions.
In fact, $(D_R^{2f_\ell\cdot\ell}\otimes D_L^{2f_\ell\cdot\ell})$ acts on the set of smooth bisections $\{X_R^{2n_\ell}(j,m_j)\times X_L^{2n_\ell}(j,m_j)\}_{j,m_j}$ of $X_R^{2n_\ell}\times X_L^{2n_\ell}$.
$D_L^{2f_\ell\cdot\ell}$ has the form $D_L^{2f_\ell\cdot\ell}(X_L^{n_\ell}(j,m_j)) = \sum\limits_a\dots\sum\limits_t \partial_1^a \dots \partial_{2f_\ell\cdot\ell}^t(X_L^{2n_\ell}(j,m_j))$, where $\partial_{2f_\ell\cdot\ell}=i\,\dfrac{d}{dx_{2f_\ell\cdot\ell}}$ is the differential operator with respect to the $2f_\ell\cdot\ell$-th variable $x_{2f_\ell\cdot\ell}$.

3.4 Definition: Symbol of the bioperator $D_R^{2f_\ell\cdot\ell}\otimes D_L^{2f_\ell\cdot\ell}$

Referring to section 2.4 and lemma 2.5, $X_R^{2n_\ell}[2f_\ell\cdot\ell]\times X_L^{2n_\ell}[2f_\ell\cdot\ell]$ develops according to:

$$X_R^{2n_\ell}[2f_\ell\cdot\ell]\times X_L^{2n_\ell}[2f_\ell\cdot\ell]=(\Delta_R^{2f_\ell\cdot\ell}\times\Delta_L^{2f_\ell\cdot\ell})\times(X_R^{2n_\ell-2f_\ell\cdot\ell}\times X_L^{2n_\ell-2f_\ell\cdot\ell})$$

where $(\Delta_R^{2f_\ell\cdot\ell}\times\Delta_L^{2f_\ell\cdot\ell})$ is the total space of the tangent bibundle $\mathrm{TAN}[SP^{f_\ell}(X_R^{\mathrm{sv}}]\times\mathrm{TAN}[SP^{f_\ell}(X_L^{\mathrm{sv}}]$ and develops according to:

$$\begin{aligned}\Delta_R^{2f_\ell\cdot\ell}\times\Delta_L^{2f_\ell\cdot\ell} &\simeq \mathrm{AdFRepsp}(\mathrm{GL}_{f_\ell\cdot\ell}(\mathbb{C}\times\mathbb{C}))\times\mathrm{FRepsp}(\mathrm{GL}_{f_\ell\cdot\ell}(F_{\overline{\omega}}\times F_\omega))\\ &=\mathrm{FRepsp}(\mathrm{GL}_{f_\ell\cdot\ell}((F_{\overline{\omega}}\times\mathbb{C})\times(F_{\overline{\omega}}\times\mathbb{C})))\ .\end{aligned}$$

Then, referring to the classical definition [A-S] of the symbol $\sigma(D)$ of a differential linear operator D, we can admit that the symbol $\sigma(D_R^{2f_\ell\cdot\ell}\otimes D_L^{2f_\ell\cdot\ell})$ of the bioperator $(D_R^{2f_\ell\cdot\ell}\otimes D_L^{2f_\ell\cdot\ell})$ can be introduced by [Ma1], [Ma2], [L-T]:

$$\sigma(D_R^{2f_\ell\cdot\ell}\otimes D_L^{2f_\ell\cdot\ell})=\mathrm{FRepsp}(P_{f_\ell\cdot\ell}((F_{\overline{\omega}}\otimes\mathbb{C})\times(F_\omega\otimes\mathbb{C})))\ ,$$

i.e. by the unitary functional representation space of $\mathrm{GL}_{f_\ell\cdot\ell}((F_{\overline{\omega}}\otimes\mathbb{C})\times(F_\omega\otimes\mathbb{C}))$ given by the functional representation space of the shifted bilinear parabolic semigroup $P_{f_\ell\cdot\ell}((F_{\overline{\omega}}\otimes\mathbb{C})\times(F_\omega\otimes\mathbb{C}))$ [Pie3].

3.5 Definition

The differential bioperator $(D_R^{2f_\ell\cdot\ell}\otimes D_L^{2f_\ell\cdot\ell})$ is elliptic if its symbol $\sigma(D_R^{2f_\ell\cdot\ell}\otimes D_L^{2f_\ell\cdot\ell})$ is invertible.

In connection with the work of G. Kasparov [Kas] who constructed a general $K_*\ K^*$ functor on the categories of compact operators and Hilbert modules, **we shall introduce a bilinear $\boldsymbol{K_*\ K^*}$ functor on the categories of elliptic bioperators and products, right by left, of Suslin-Voevodsky pure semimotives allowing to set up a bilinear version of the index theorem**[B-F-M], [A-H], [Mil], [Jan1].

3.6 Chern character of the pure bimotive $\underline{c}_*(X^{\rm sv}_{R\times L})$

Let $H^{2n_\ell}(\underline{c}_*(X^{\rm sv}_{R\times L}),X^{2n_\ell}_R\times X^{2n_\ell}_L)$ be the bilinear cohomology of the Suslin-Voevodsky pure bisemimotive $\underline{c}_*(X^{\rm sv}_{R\times L})$ and let

$$H^*(\underline{c}_*(X^{\rm sv}_{R\times L}))=\bigoplus_{n_\ell} H^{2n_\ell}(\underline{c}_*(X^{\rm sv}_{R\times L}),X^{2n_\ell}_R\times X^{2n_\ell}_L)$$

denote the total bilinear cohomology of $\underline{c}_*(X^{\rm sv}_{R\times L})$.
Taking into account the definition of a pure bisemimotive $\underline{c}_*(X^{\rm sv}_{R\times L})$ as being a functor from $X^{\rm sv}_{R\times L}$ of complex dimension ℓ to the chain bicomplex associated to the product, right by left, of abelian semigroups

$$\bigsqcup_{i_\ell} \mathrm{Hom}_{Sm_L(k)\times Sm_R(k)}(\dot\Sigma_R\times\dot\Sigma_L,SP^{i_\ell}(X^{\rm sv}_R)\times SP^{i_\ell}(X^{\rm sv}_L))\ ,$$

we can introduce the product, right by left, of abelian semigroups generated by the complex vector bundles [Laf] over $X^{\rm sv}_R\times X^{\rm sv}_L$ and noted $K^*(X^{\rm sv}_{R\times L})$
$K^*(X^{\rm sv}_{R\times L})$ is then the K-cohomology associated to the pure bisemimotive $\underline{c}_*(X^{\rm sv}_{R\times L})$.

The total Chern character [Gil2] in the bilinear $\boldsymbol{K}$-cohomology of the pure bimotive $\underline{c}_*(X^{\rm sv}_{R\times L})$ is then given by the homomorphism:

$$ch^*(\underline{c}_*(X^{\rm sv}_{R\times L})):\qquad K^*(X^{\rm sv}_{R\times L}))\longrightarrow H^*(\underline{c}_*(X^{\rm sv}_{R\times L}))$$

and defined, as classically, according to:

$$ch^*(\underline{c}_*(X^{\rm sv}_{R\times L}))=\sum_{i_\ell=1}^{n_\ell/\ell}e^{\gamma_{i_\ell}}\centerdot e^{\gamma_{i_\ell}}\ ,\qquad i_\ell\centerdot\ell=n_\ell\le n$$

where the γ_{i_ℓ} result from the factorization [Hir]:

$$(1+c_1\ x+\cdots+c_{i_\ell}\ x^{i_\ell}+\cdots+c_{n_\ell/\ell}\ x^{n_\ell/\ell})=\prod_{i_\ell}(1+\gamma_{i_\ell}\ x)$$

with the c_{i_ℓ} being the Chern classes.

3.7 Bilinear K-homology [B-D-F], [Blo], [Gil1]

- Let $H_{2f_\ell\cdot\ell}(\Delta_R^{2f_\ell\cdot\ell}\times\Delta_L^{2f_\ell\cdot\ell}),\mathcal{F}^*_{R\times L}(\mathrm{TAN}))\simeq\mathrm{AdFRepsp}(\mathrm{GL}_{f_\ell\cdot\ell}(\mathbb{C}\times\mathbb{C}))$ be the homology with coefficients in the bilinear fiber $\mathcal{F}^{2f_\ell\cdot\ell}_{R\times L}(\mathrm{TAN})$ of the tangent bibundle $\mathrm{TAN}[SP^{f_\ell}(X_R^{\mathrm{sv}})\times SP^{f_\ell}(X_L^{\mathrm{sv}})]$ and let

$$\begin{aligned}H_*(\Delta_R^*\times\Delta_L^*,\mathcal{F}^*_{R\times L}(\mathrm{TAN})&=\bigoplus_{f_\ell\cdot\ell}H_{2f_\ell\cdot\ell}(\Delta_R^{2f_\ell\cdot\ell}\times\Delta_L^{2f_\ell\cdot\ell}),\mathcal{F}^{2f_\ell\cdot\ell}_{R\times L}(\mathrm{TAN}))\\ &\simeq\bigoplus_{f_\ell\cdot\ell}\mathrm{AdFRepsp}(\mathrm{GL}_{f_\ell\cdot\ell}(\mathbb{C}\times\mathbb{C}))\end{aligned}$$

  be the total bilinear homology with coefficients in the set of bilinear fibres $\mathcal{F}^{2f_\ell\cdot\ell}_{R\times L}(\mathrm{TAN})$ such that, for $i_\ell\times\ell=n_\ell=(f_\ell\cdot\ell)+(n_\ell-f_\ell\cdot\ell)$, $f_\ell\leq i_\ell$ and $f_\ell\cdot\ell\leq n_\ell\leq n$.

- Then, a **bilinear K-homology**, noted $K_*(SP^{FL}(X^{\mathrm{sv}}_{R\times L}))$, can be introduced as being the product, right by left, of abelian semigroups generated by the set of tangent bibundles $\mathrm{TAN}[SP^{f_\ell}(X_R^{\mathrm{sv}})\times SP^{f_\ell}(X_L^{\mathrm{sv}})]$, for all $f_\ell\cdot\ell\leq n_\ell\leq n$, on the product $SP^{f_\ell}(X_R^{\mathrm{sv}})\times SP^{f_\ell}(X_L^{\mathrm{sv}})$ of smooth presheaves.

  $K_*(SP^{FL}(X^{\mathrm{sv}}_{R\times L}))$ is the K-homology associated to the pure bisemimotive $\underline{c}_*(X^{\mathrm{sv}}_{R\times L})$.

- **The Chern character in this bilinear K-homology** is thus given by the homomorphism:

$$ch_*(\underline{c}_*(X^{\mathrm{sv}}_{R\times L})):\quad K_*(SP^{FL}(X^{\mathrm{sv}}_{R\times L}))\longrightarrow H_*(\Delta_R^*\times\Delta_L^*,\mathcal{F}^*_{R\times L}(\mathrm{TAN}))\;.$$

  Taking into account that $H_*(\Delta_R^*\times\Delta_L^*,\mathcal{F}^*_{R\times L}(\mathrm{TAN}))$ is the homology of $(\Delta_R^*\times\Delta_L^*)$ with coefficients in the set of bilinear fibres $\mathcal{F}^{2f_\ell\cdot\ell}_{R\times L}(\mathrm{TAN})$ which are "contracting", the total Chern character in this bilinear K-homology will be defined by

$$ch_*(\underline{c}_*(X^{\mathrm{sv}}_{R\times L}))=\sum_{f_\ell}e^{-\gamma_{f_\ell}}\centerdot e^{-\gamma_{f_\ell}}$$

  such that the γ_{f_ℓ} are obtained from a formal factorisation $\sum\limits_{f_\ell}c_{-f_\ell}\;x^{f_\ell}=\prod\limits_{f_\ell}(1-\gamma_{f_\ell}\;x)$ where the $c_{-f_\ell}\in H^{-2f_\ell\cdot\ell}(\mathcal{F}^{f_\ell\cdot\ell}_{\{^R_L}(\mathrm{TAN}),\mathbb{Z})$ are Chern classes associated with the homology.

3.8 Proposition

The total Chern character $ch^*(M_{DM_{R\times L}}(X^{\mathrm{sv}}_{R\times L}))$ of the Suslin-Voevodsky mixed bisemimotive $M_{DM_{R\times L}}(X^{\mathrm{sv}}_{R\times L})$ in the mixed bilinear K-homology-K-cohomology is given by **the**

homomorphism:

$$\begin{aligned} ch^*(M_{DM_{R\times L}}(X^{\rm sv}_{R\times L})): \quad & K_*(SP^{FL}(X^{\rm sv}_{R\times L}))\times K^*(X^{\rm sv}_{R\times L}) \\ & \longrightarrow H_*(\Delta^*_R\times\Delta^*_L,\mathcal{F}^*_{R\times L}(\mathrm{TAN}))\times H^*(\underline{c}_*(X^{\rm sv}_{R\times L})) \end{aligned}$$

such that:

$$ch^*(M_{DM_{R\times L}}(X^{\rm sv}_{R\times L})) = ch_*(\underline{c}_*(X^{\rm sv}_{R\times L}))\times ch^*(\underline{c}_*(X^{\rm sv}_{R\times L}))$$

corresponds to a bilinear version of the index theorem.

Proof. Taking into account that:

$$H^*(\underline{c}_*(X^{\rm sv}_{R\times L})) = \bigoplus_{n_\ell} H^{2n_\ell}(\underline{c}_*(X^{\rm sv}_{R\times L}),X_R^{2n_\ell}\times X_L^{2n_\ell})$$

and that:

$$H_*(\Delta^*_R\times\Delta^*_L,\mathcal{F}^*_{R\times L}(\mathrm{TAN})) = \bigoplus_{f_\ell\cdot\ell} H_{2f_\ell\cdot\ell}(\Delta_R^{2f_\ell\cdot\ell}\times\Delta_L^{2f_\ell\cdot\ell},\mathcal{F}^{2f_\ell\cdot\ell}_{R\times L}(\mathrm{TAN}))$$

according to sections 3.6 and 3.7, as well as the decomposition of the cohomology of the Suslin-Voevodsky mixed bimotive into:

$$\begin{aligned} & H^{2n_\ell-2f_\ell\cdot\ell}(M_{DM_{R\times L}}(X^{\rm sv}_{R\times L}),X_R^{2n_\ell}[2f_\ell\cdot\ell]\times X_L^{2n_\ell}[2f_\ell\cdot\ell]) \\ & \qquad \longrightarrow H_{2f_\ell\cdot\ell}(\Delta_R^{2f_\ell\cdot\ell}\times\Delta_L^{2f_\ell\cdot\ell},\mathcal{F}^{2f_\ell\cdot\ell}_{R\times L}(\mathrm{TAN}))\times H^{2n_\ell}(\underline{c}_*(X^{\rm sv}_{R\times L}),X_R^{2n_\ell}\times X_L^{2n_\ell})\ , \end{aligned}$$

we have that:

$$\begin{aligned} & H_*(\Delta^*_R\times\Delta^*_L)\times H^*(\underline{c}_*(X^{\rm sv}_{R\times L})) \\ & \qquad = \bigoplus_{n_\ell}\bigoplus_{f_\ell\cdot\ell} H^{2n_\ell-2f_\ell\cdot\ell}(M_{DM_{R\times L}}(X^{\rm sv}_{R\times L}),X_R^{2n_\ell}[2f_\ell\cdot\ell]\times X_L^{2n_\ell}[2f_\ell\cdot\ell]) \end{aligned}$$

is the total bilinear cohomology of $M_{DM_{R\times L}}(X^{\rm sv}_{R\times L})$, noted $H^*(M_{DM_{R\times L}}(X^{\rm sv}_{R\times L})$.
Similar arguments can be used to prove that

$$K^*(M_{DM_{R\times L}}(X^{\rm sv}_{R\times L})) = K_*(SP^{FL}(X^{\rm sv}_{R\times L}))\times K^*(X^{\rm sv}_{R\times L})$$

is the mixed bilinear K-homology-K-cohomology associated with the Suslin-Voevodsky mixed bisemimotive $M_{DM_{R\times L}}(X^{\rm sv}_{R\times L})$.
And, thus, it follows that:

$$\begin{aligned} ch^*(M_{DM_{R\times L}}(X^{\rm sv}_{R\times L})): \quad & K^*(M_{DM_{R\times L}}(X^{\rm sv}_{R\times L})) \\ & \longrightarrow H^*(M_{DM_{R\times L}}(X^{\rm sv}_{R\times L})) \end{aligned}$$

is the total Chern character of the Suslin-Voevodsky mixed bisemimotive. ∎

3.9 Corollary

Let $(D_R^{2f_\ell\cdot\ell}\otimes D_L^{2f_\ell\cdot\ell})$ *be a differential bioperator defined by its biaction:*

$$D_R^{2f_\ell\cdot\ell}\otimes D_L^{2f_\ell\cdot\ell}: \qquad X_R^{2n_\ell}\times X_L^{2n_\ell} \longrightarrow X_R^{2n_\ell}[2f_\ell\cdot\ell]\times X_L^{2n_\ell}[2f_\ell\cdot\ell]$$

from the Suslin-Voevodsky pure subbisemimotive $X_R^{2n_\ell}\times X_L^{2n_\ell}$ *to the corresponding mixed subbimotive* $X_R^{2n_\ell}[2f_\ell\cdot\ell]\times X_L^{2n_\ell}[2f_\ell\cdot\ell]$ *.*
Let

$$ch_*(D_R^{2f_\ell\cdot\ell}\otimes D_L^{2f_\ell\cdot\ell}): \quad K_*(SP^{f_\ell}(X^{\mathrm{sv}}_{R\times L})) \longrightarrow H_{2f_\ell\cdot\ell}(\Delta_R^{2f_\ell\cdot\ell}\times\Delta_L^{2f_\ell\cdot\ell},\mathcal{F}_{R\times L}^{2f_\ell\cdot\ell}(\mathrm{TAN}))$$

be an element of the Chern character $ch_*(\underline{c}_*(X^{\mathrm{sv}}_{R\times L}))$ *associated with the biaction of* $(D_R^{2f_\ell\cdot\ell}\otimes D_L^{2f_\ell\cdot\ell})$ *on* $(X_R^{2n_\ell}\times X_L^{2n_\ell})$ *.*
Let

$$ch^*(X_R^{2n_\ell}\times X_L^{2n_\ell}): \quad K^*(SP^{f_\ell}(X^{\mathrm{sv}}_{R\times L})) \longrightarrow H^{2n_\ell}(\underline{c}_*(X^{\mathrm{sv}}_{R\times L}),X_R^{2n_\ell}\times X_L^{2n_\ell})$$

denote an element of $ch^*(\underline{c}_*(X^{\mathrm{sv}}_{R\times L}))$ *.*
Then, $ch_*(D_R^{2f_\ell\cdot\ell}\otimes D_L^{2f_\ell\cdot\ell})\times ch^*(X_R^{2n_\ell}\times X_L^{2n_\ell})$ *will allow to define an index* $\mathrm{Ind}(D_R^{2f_\ell\cdot\ell}\otimes D_L^{2f_\ell\cdot\ell})$ *of the elliptic bioperator which is different from the classical Atiyah-Singer index* $\gamma(D_R^{2f_\ell\cdot\ell}\otimes D_L^{2f_\ell\cdot\ell})$ *except if* $i_\ell=f_\ell$ *.*

Proof. Referring to sections 3.6 and 3.7 where

$$ch^*(\underline{c}_*(X^{\mathrm{sv}}_{R\times L})) = \sum_{i_\ell=1}^{n_\ell/\ell} e^{\gamma_{i_\ell}}\centerdot e^{\gamma_{i_\ell}}\ , \qquad\qquad i_\ell\centerdot\ell=n_\ell\leq n\ ,$$

$$\text{and}\qquad ch_*(\underline{c}_*(X^{\mathrm{sv}}_{R\times L})) = \sum_{f_\ell} e^{-\gamma_{f_\ell}}\centerdot e^{-\gamma_{f_\ell}}\ ,$$

are introduced, we define $\mathrm{Ind}(D_R^{2f_\ell\cdot\ell}\otimes D_L^{2f_\ell\cdot\ell})$ by:

$$\begin{aligned}\mathrm{Ind}(D_R^{2f_\ell\cdot\ell}\otimes D_L^{2f_\ell\cdot\ell}) &= ch_*(D_R^{2f_\ell\cdot\ell}\otimes D_L^{2f_\ell\cdot\ell})\times ch^*(X_R^{2n_\ell}\times X_L^{2n_\ell})\\ &= e^{2\gamma_{i_\ell}}\centerdot e^{-2\gamma_{f_\ell}}-\delta_{i_\ell,f_\ell}\ ,\end{aligned}$$

where $\delta_{i_\ell,f_\ell}=0$ if $i_\ell\neq f_\ell$,

$\qquad\qquad =1$ if $i_\ell=f_\ell$.

Thus, $\mathrm{Ind}(D_R^{2f_\ell\cdot\ell}\otimes D_L^{2f_\ell\cdot\ell})=0$ if and only if $i_\ell=f_\ell$. In that case, if $(D_R^{2f_\ell\cdot\ell}\otimes D_L^{2f_\ell\cdot\ell})$ is of finite rank, $\mathrm{Ind}(D_R^{2f_\ell\cdot\ell}\otimes D_L^{2f_\ell\cdot\ell})=0$ and could correspond to the classical Atiyah-Singer index, defined by

$$\gamma(D_R^{2f_\ell\cdot\ell}\otimes D_L^{2f_\ell\cdot\ell}) = \dim\mathrm{Ker}(D_R^{2f_\ell\cdot\ell}\otimes D_L^{2f_\ell\cdot\ell})-\dim\mathrm{coKer}(D_R^{2f_\ell\cdot\ell}\otimes D_L^{2f_\ell\cdot\ell})\ . \qquad \blacksquare$$

3.10 Corollary

The equivalent of the classical index theorem [Gil1] *for a Suslin-Voevodsky bisemimotive asserts that, if*

$$\{D_R^{2f_\ell\cdot\ell}\otimes D_L^{2f_\ell\cdot\ell}\}_{f_\ell\cdot\ell}: \qquad \underline{c}_*(X^{\rm sv}_{R\times L}) \longrightarrow M_{DM_{R\times L}}(X^{\rm sv}_{R\times L})$$

is a proper morphism from the Suslin-Voevodsky pure bisemimotive $\underline{c}_*(X^{\rm sv}_{R\times L})$ *to the Suslin-Voevodsky mixed bisemimotive* $M_{DM_{R\times L}}(X^{\rm sv}_{R\times L})$ *under the action of the set of bioperators* $\{D_R^{2f_\ell\cdot\ell}\otimes D_L^{2f_\ell\cdot\ell}\}_{f_\ell\cdot\ell}$*, we have that:*

$$\begin{aligned} ch_*(\underline{c}_*(X^{\rm sv}_{R\times L}))\times ch^*(\underline{c}_*(X^{\rm sv}_{R\times L})) &= ch^*(M_{DM_{R\times L}}(X^{\rm sv}_{R\times L}))\\ &= \mathrm{Ind}\{(D_R^{2f_\ell\cdot\ell}\otimes D_L^{2f_\ell\cdot\ell})\}_{f_\ell\cdot\ell}\ . \end{aligned}$$

Proof. More specifically, the index theorem would assert that:

$$\mathrm{Im}\{D_R^{2f_\ell\cdot\ell}\otimes D_L^{2f_\ell\cdot\ell}\}_{f_\ell\cdot\ell} ch^*(\underline{c}_*(X^{\rm sv}_{R\times L})) = ch^*(M_{DM_{R\times L}}(X^{\rm sv}_{R\times L}))$$

where $\mathrm{Im}\{D_R^{2f_\ell\cdot\ell}\otimes D_L^{2f_\ell\cdot\ell}\}_{f_\ell\cdot\ell}$ is the image of the morphism generated by $\{D_R^{2f_\ell\cdot\ell}\otimes D_L^{2f_\ell\cdot\ell}\}_{f_\ell\cdot\ell}$.
Now, if we take into account the considered notations, it appears that:

$$\mathrm{Im}\{D_R^{2f_\ell\cdot\ell}\otimes D_L^{2f_\ell\cdot\ell}\}_{f_\ell\cdot\ell}(ch^*(\underline{c}_*(X^{\rm sv}_{R\times L}))) = ch_*(\underline{c}_*(X^{\rm sv}_{R\times L}))\times ch^*(\underline{c}_*(X^{\rm sv}_{R\times L}))$$

since $\mathrm{Im}\{D_R^{2f_\ell\cdot\ell}\otimes D_L^{2f_\ell\cdot\ell}\}_{f_\ell\cdot\ell}$ is generated by the set of bioperators $\{D_R^{2f_\ell\cdot\ell}\otimes D_L^{2f_\ell\cdot\ell}\}_{f_\ell\cdot\ell}$ to which the Chern character $ch_*(\underline{c}_*(X^{\rm sv}_{R\times L}))$ in this bilinear K-homology corresponds.
Furthermore, we have that:

$$\begin{aligned} \mathrm{Ind}\{(D_R^{2f_\ell\cdot\ell}\otimes D_L^{2f_\ell\cdot\ell})\}_{f_\ell\cdot\ell} &= ch_*(\underline{c}_*(X^{\rm sv}_{R\times L}))\times ch^*(\underline{c}_*(X^{\rm sv}_{R\times L}))\\ &= ch_*(M_{DM_{R\times L}}(X^{\rm sv}_{R\times L}))\ . \end{aligned}$$

■

4 The toroidal spectral representation of an elliptic bioperator

Chapters 2 and 3 were essentially devoted to pure and mixed bimotives of Suslin-Voevodsky, while chapter 4 and 5 will more particularly concern the functional representation spaces of bilinear algebraic semigroups.

4.1 The shifted compactified bisemispace $\overline{S}^{P_{n_\ell[f_\ell\cdot\ell]}}_{\mathrm{GL}_{n_\ell[f_\ell\cdot\ell]}}$

- Let
$$\overline{S}^{P_{n_\ell}}_{\mathrm{GL}_{n_\ell}}=P_{n_\ell}(F^T_{\overline{\omega}^1_1}\times F^T_{\omega^1_1})\setminus \mathrm{GL}_{n_\ell}(F^T_R\times F^T_L)\big/\,\mathrm{GL}_{n_\ell}((\mathbb{Z}\,/N\,\mathbb{Z}\,)^2)$$
be the toroidal compactified bisemispace representing the double coset decomposition of the algebraic bilinear semigroup $\mathrm{GL}_{n_\ell}(F^T_R\times F^T_L)$ as introduced in section 1.12.

- The corresponding **double coset decomposition of the bilinear general semigroup shifted in $(f_\ell\cdot\ell\times f_\ell\cdot\ell)$ complex dimensions** $\mathrm{GL}_{n_\ell[f_\ell\cdot\ell]}(F^T_R\otimes\mathbb{C}\,)\times(F^T_L\otimes\mathbb{C}\,))$, as developed in proposition 2.12, is given by:
$$\begin{aligned}\overline{S}^{P_{n_\ell[f_\ell\cdot\ell]}}_{\mathrm{GL}_{n_\ell[f_\ell\cdot\ell]}}&=P_{n_\ell[f_\ell\cdot\ell]}((F^T_{\overline{\omega}^1}\otimes\mathbb{C}\,)\times(F^T_{\omega^1}\otimes\mathbb{C}\,))\\&\qquad\setminus \mathrm{GL}_{n_\ell[f_\ell\cdot\ell]}((F^T_R\otimes\mathbb{C}\,)\times(F^T_L\otimes\mathbb{C}\,))\big/\,\mathrm{GL}_{n_\ell[f_\ell\cdot\ell]}((\mathbb{Z}\,/N\,\mathbb{Z}\,)^2\otimes\mathbb{C}^{\,2})\end{aligned}$$
in such a way that:

  1) $\overline{S}^{P_{n_\ell[f_\ell\cdot\ell]}}_{\mathrm{GL}_{n_\ell[f_\ell\cdot\ell]}}=\mathrm{Repsp}(\mathrm{GL}_{n_\ell[f_\ell\cdot\ell]}((F^T_{\overline{\omega}}\otimes\mathbb{C}\,)\times(F^T_{\omega}\otimes\mathbb{C}\,)))$ implies that
$$\begin{aligned}&\mathrm{FRepsp}(\mathrm{GL}_{n_\ell[f_\ell\cdot\ell]}((F^T_{\overline{\omega}}\otimes\mathbb{C}\,)\times(F^T_{\omega}\otimes\mathbb{C}\,)))\\&\qquad=\mathrm{AdFRepsp}(\mathrm{GL}_{f_\ell\cdot\ell}((\mathbb{C}\,\otimes\mathbb{C}\,)\times\mathrm{FRepsp}(\mathrm{GL}_{n_\ell}(F^T_{\overline{\omega}}\otimes F^T_{\omega}))\end{aligned}$$
according to proposition 2.12.

  2) The **shifted complex bilinear parabolic semigroup** $P_{n_\ell[f_\ell\cdot\ell]}((F^T_{\overline{\omega}^1}\otimes\mathbb{C}\,)\times(F^T_{\omega^1}\otimes\mathbb{C}\,)$ is generated from its unshifted equivalent $P_{n_\ell}((F^T_{\overline{\omega}^1}\times F^T_{\omega^1})$ by the shift homomorphism:
$$SH_{P_{n_\ell}}:\qquad P_{n_\ell}(F^T_{\overline{\omega}^1}\times F^T_{\omega^1})\longrightarrow P_{n_\ell[f_\ell\cdot\ell]}((F^T_{\overline{\omega}^1}\otimes\mathbb{C}\,)\times(F^T_{\omega^1}\otimes\mathbb{C}\,))\,.$$
A similar shift homomorphism can be introduced in order to generate $\mathrm{GL}_{n_\ell[f_\ell\cdot\ell]}((F^T_{\overline{\omega}}\otimes\mathbb{C}\,)\times(F^T_{\omega}\otimes\mathbb{C}\,))$.

3) The bilinear arithmetic subgroup $\mathrm{GL}_{n_\ell}((\mathbb{Z}/N\,\mathbb{Z})^2)$, generating a $(\mathbb{Z}/N\,\mathbb{Z})^2$-bilattice in $\overline{S}^{P_{n_\ell}}_{\mathrm{GL}_{n_\ell}}$, is transformed by the shift homomorphism:

$$SH_{\mathrm{GL}_{n_\ell}}: \qquad \mathrm{GL}_{n_\ell}((\mathbb{Z}/N\,\mathbb{Z})^2) \longrightarrow \mathrm{GL}_{n_\ell[f_\ell\cdot\ell]}((\mathbb{Z}/N\,\mathbb{Z})^2)\otimes\mathbb{C}^2)$$

into a **shifted bilinear arithmetic subgroup** $\mathrm{GL}_{n_\ell[f_\ell\cdot\ell]}((\mathbb{Z}/N\,\mathbb{Z})^2)\otimes\mathbb{C}^2)$ in such a way that:

- the (functional) representation space of $\mathrm{GL}_{n_\ell[f_\ell\cdot\ell]}((\mathbb{Z}/N\,\mathbb{Z})^2)\otimes\mathbb{C}^2)$ corresponds to the Lie algebra of $\mathrm{GL}_{n_\ell}((\mathbb{Z}/N\,\mathbb{Z})^2))$:

  $$\mathrm{Lie}(\mathrm{GL}_{n_\ell}(\mathbb{Z}/N\,\mathbb{Z})^2)\approx \mathrm{FRepsp}(\mathrm{GL}_{n_\ell[f_\ell\cdot\ell]}((\mathbb{Z}/N\,\mathbb{Z})^2)\otimes\mathbb{C}^2)$$

  by considerations similar as given in proposition 2.10.
- a shifted pseudoramified Hecke bialgebra $\mathcal{H}_{R\times L}(n_\ell[f_\ell\cdot\ell])$, generated by the shifted pseudoramified Hecke bioperators $T_R(n_\ell[f_\ell\cdot\ell];t)\otimes T_L(n_\ell[f_\ell\cdot\ell];t)$, has a representation in $\mathrm{GL}_{n_\ell[f_\ell\cdot\ell]}((\mathbb{Z}/N\,\mathbb{Z})^2)\otimes\mathbb{C}^2)$ as developed in the next proposition.

4.2 Proposition

Let $\mathrm{GL}_{n_\ell[f_\ell\cdot\ell]}((\mathbb{Z}/N\,\mathbb{Z})^2\otimes\mathbb{C}^2)$ *be the shifted bilinear arithmetic subgroup generated from* $\mathrm{GL}_{n_\ell}((\mathbb{Z}/N\,\mathbb{Z})^2)$.
Then, the pseudoramified Hecke bialgebra, generated by all the shifted pseudoramified Hecke bioperators $T_R(2n_\ell[2f_\ell\cdot\ell];t)\otimes T_L(2n_\ell[2f_\ell\cdot\ell];t)$, *is a shifted pseudoramified bialgebra of Hecke noted* $\mathcal{H}_{R\times L}(2n_\ell[2f_\ell\cdot\ell])$.

Proof. 1) Referring to sections 1.4 and 1.8, a shifted maximal order of F^T_ω (resp. $F^T_{\overline{\omega}}$) will be given by $(\mathcal{O}_{F^T_\omega}\otimes\mathbb{C})$ (resp. $(\mathcal{O}_{F^T_{\overline{\omega}}}\otimes\mathbb{C})$).

Then, a lattice of dimension $2n_\ell$, noted $\Lambda^{2n_\ell}_\omega$ (resp. $\Lambda^{2n_\ell}_{\overline{\omega}}$) shifted in $2f_\ell\cdot\ell$ dimensions will be introduced by:

$$\Lambda^{2n_\ell[2f_\ell\cdot\ell]}_\omega=\Lambda^{2n_\ell}_\omega\otimes_{[f_\ell\cdot\ell]}\mathbb{C}\ , \qquad (\text{resp.}\quad \Lambda^{2n_\ell[2f_\ell\cdot\ell]}_{\overline{\omega}}=\Lambda^{2n_\ell}_{\overline{\omega}}\otimes_{[f_\ell\cdot\ell]}\mathbb{C}\ ),$$

where the tensor product $\otimes_{[f_\ell\cdot\ell]}$ bears on the $f_\ell\cdot\ell$ shifted complex dimensions, and will be defined by the isomorphism:

$$\begin{aligned}
\Lambda^{2n_\ell}_\omega\otimes_{[f_\ell\cdot\ell]}\mathbb{C} &\simeq T_{n_\ell[f_\ell\cdot\ell]}((\mathbb{Z}/N\,\mathbb{Z})\otimes\mathbb{C})\\
&= T_{n_\ell[f_\ell\cdot\ell]}(\mathcal{O}_{F^T_\omega}\otimes\mathbb{C})\\
(\text{resp.}\quad \Lambda^{2n_\ell}_{\overline{\omega}}\otimes_{[f_\ell\cdot\ell]}\mathbb{C} &\simeq T^t_{n_\ell[f_\ell\cdot\ell]}((\mathbb{Z}/N\,\mathbb{Z})\otimes\mathbb{C})\\
&= T^t_{n_\ell[f_\ell\cdot\ell]}(\mathcal{O}_{F^T_{\overline{\omega}}}\otimes\mathbb{C})\ )
\end{aligned}$$

leading to:

$$\Lambda_{\overline{\omega}}^{2n_\ell[2f_\ell\cdot\ell]}\otimes\Lambda_{\omega}^{2n_\ell[2f_\ell\cdot\ell]}\simeq \mathrm{GL}_{n_\ell[f_\ell\cdot\ell]}((\mathbb{Z}/N\,\mathbb{Z})^2)\otimes\mathbb{C}^2)\ .$$

2) According to proposition 1.10, the (j,m_j)-th coset representative $U_{j,m_{j_R}}(2n_\ell[2f_\ell\cdot\ell])\times U_{j,m_{j_L}}(2n_\ell[2f_\ell\cdot\ell])$ of the shifted pseudoramified Hecke bioperator $T_R(2n_\ell[2f_\ell\cdot\ell];t)\otimes T_L(2n_\ell[2f_\ell\cdot\ell];t)$ will be given by:

$$\begin{aligned}
U_{j,m_{j_R}}&(2n_\ell[2f_\ell\cdot\ell])\times U_{j,m_{j_L}}(2n_\ell[2f_\ell\cdot\ell])\\
&=[d_{n_\ell[f_\ell\cdot\ell]}((\mathbb{Z}/N\,\mathbb{Z})\otimes\mathbb{C})\times d_{n_\ell[f_\ell\cdot\ell]}((\mathbb{Z}/N\,\mathbb{Z})\otimes\mathbb{C})]\\
&\qquad\times[u^t_{n_\ell[f_\ell\cdot\ell]}((\mathbb{Z}/N\,\mathbb{Z})\otimes\mathbb{C})\times u_{n_\ell[f_\ell\cdot\ell]}((\mathbb{Z}/N\,\mathbb{Z})\otimes\mathbb{C})]
\end{aligned}$$

taking into account the Gauss bilinear decomposition. ■

4.3 Proposition

The differential bioperator $(D_R^{2f_\ell\cdot\ell}\otimes D_L^{2f_\ell\cdot\ell})\in\mathcal{D}_R\otimes\mathcal{D}_L$ *maps the bisemisheaf* $(\widehat{M}^{2n_\ell}_{T_R}\otimes\widehat{M}^{2n_\ell}_{T_L})$ *into the corresponding perverse bisemisheaf* $(\widehat{M}^{2n_\ell}_{T_R}[2f_\ell\cdot\ell]\otimes\widehat{M}^{2n_\ell}_{T_L}[2f_\ell\cdot\ell])$ *according to:*

$$D_R^{2f_\ell\cdot\ell}\otimes D_L^{2f_\ell\cdot\ell}:\qquad(\widehat{M}^{2n_\ell}_{T_R}\otimes\widehat{M}^{2n_\ell}_{T_L})\longrightarrow(\widehat{M}^{2n_\ell}_{T_R}[2f_\ell\cdot\ell]\otimes\widehat{M}^{2n_\ell}_{T_L}[2f_\ell\cdot\ell])$$

such that $(\widehat{M}^{2n_\ell}_{T_R}[2f_\ell\cdot\ell]\otimes\widehat{M}^{2n_\ell}_{T_L}[2f_\ell\cdot\ell])$ *is the tensor product of perverse (semi)sheaves which are* $\mathcal{D}_R\otimes\mathcal{D}_L$*-bisemimodules.*

Proof. 1) The action of the differential bioperator $(D_R^{2f_\ell\cdot\ell}\otimes D_L^{2f_\ell\cdot\ell})$ on $\widehat{M}^{2n_\ell}_{T_R}\otimes\widehat{M}^{2n_\ell}_{T_L}$ corresponds to the shift homomorphism:

$$SH_{G_{n_\ell}}:\qquad\mathrm{GL}_{n_\ell}(F^T_{\overline{\omega}}\times F^T_{\omega})\longrightarrow\mathrm{GL}_{n_\ell[f_\ell\cdot\ell]}((F^T_{\overline{\omega}}\otimes\mathbb{C})\times(F^T_{\omega}\otimes\mathbb{C}))\ ,$$

as introduced in section 4.1, since $M^{2n_\ell}_{T_R}\otimes M^{2n_\ell}_{T_L}$ is the representation space of $\mathrm{GL}_{n_\ell}(F^T_{\overline{\omega}}\times F^T_{\omega})$ (see section 1.14 and [Pie3]).

2) $\widehat{M}^{2n_\ell}_{T_R}[2f_\ell\cdot\ell]\otimes\widehat{M}^{2n_\ell}_{T_L}[2f_\ell\cdot\ell]$ is the tensor product of perverse sheaves because it is an object of the derived category of $(\overline{S}^{P_{n_\ell}}_{\mathrm{GL}_{n_\ell}})$ (see sections 2.4, 1.14 and [B-B-D]). ■

4.4 Proposition

The bilinear cohomology of the shifted compactified bisemispace $\overline{S}^{P_{n[f_\ell\cdot\ell]}}_{\mathrm{GL}\,n[f_\ell\cdot\ell]}$ *is* $H^{2n_\ell-2f_\ell\cdot\ell}(\overline{S}^{P_{n[f_\ell\cdot\ell]}}_{\mathrm{GL}\,n[f_\ell\cdot\ell]},\widehat{M}^{2n_\ell}_{T_R}[2f_\ell\cdot\ell]\otimes\widehat{M}^{2n_\ell}_{T_L}[2f_\ell\cdot\ell])$ *and is isomorphic to the bilinear cohomology of the Suslin-Voevodsky mixed bimotive* $M_{DM_{R\times L}}(X^{\mathrm{sv}}_{R\times L})$*, noted*

$$H^{2n_\ell-2f_\ell\cdot\ell}(M_{DM_{R\times L}}(X^{\mathrm{sv}}_{R\times L}),X^{2n_\ell}_R[2f_\ell\cdot\ell]\times X^{2n_\ell}_L[2f_\ell\cdot\ell])\ .$$

Proof. The isomorphism:

$$H^{2n_\ell-2f_\ell\cdot\ell}(M_{DM_{R\times L}}(X^{\rm sv}_{R\times L}),X_R^{2n_\ell}[2f_\ell\cdot\ell]\times X_L^{2n_\ell}[2f_\ell\cdot\ell]) \simeq H^{2n_\ell-2f_\ell\cdot\ell}(\overline{S}^{P_{n[f_\ell\cdot\ell]}}_{\mathrm{GL}_{n[f_\ell\cdot\ell]}},\widehat{M}_R^{2n_\ell}[2f_\ell\cdot\ell]\otimes\widehat{M}_L^{2n_\ell}[2f_\ell\cdot\ell])$$

results from the isomorphisms

$$H^{2n_\ell-2f_\ell\cdot\ell}(M_{DM_{R\times L}}(X^{\rm sv}_{R\times L}),X_R^{2n_\ell}[2f_\ell\cdot\ell]\times X_L^{2n_\ell}[2f_\ell\cdot\ell]) \simeq \mathrm{FRepsp}(\mathrm{GL}_{n_\ell[f_\ell\cdot\ell]}((F^T_{\overline{\omega}}\otimes\mathbb{C})\times(F^T_\omega\otimes\mathbb{C})))$$

and

$$H^{2n_\ell-2f_\ell\cdot\ell}(\overline{S}^{P_{n[f_\ell\cdot\ell]}}_{\mathrm{GL}\,n[f_\ell\cdot\ell]},\widehat{M}_R^{2n_\ell}[2f_\ell\cdot\ell]\otimes\widehat{M}_L^{2n_\ell}[2f_\ell\cdot\ell]) \simeq \mathrm{FRepsp}(\mathrm{GL}_{n_\ell[f_\ell\cdot\ell]}((F^T_{\overline{\omega}}\otimes\mathbb{C})\times(F^T_\omega\otimes\mathbb{C})))\ . \qquad \blacksquare$$

4.5 Proposition

The bilinear cohomology

$$\begin{aligned}H^{2n_\ell-2f_\ell\cdot\ell}(\overline{S}^{P_{n[f_\ell\cdot\ell]}}_{\mathrm{GL}_{n[f_\ell\cdot\ell]}},\widehat{M}_{T_R}^{2n_\ell}[2f_\ell\cdot\ell]\otimes\widehat{M}_{T_L}^{2n_\ell}[2f_\ell\cdot\ell]) &\simeq \mathrm{CY}^{2n_\ell}(Y_R,[2f_\ell\cdot\ell])\times\mathrm{CY}^{2n_\ell}(Y_L,[2f_\ell\cdot\ell])\\ &= \{(\mathrm{CY}^{2n_\ell}(Y_R,[2f_\ell\cdot\ell],(j,m_j))\times(\mathrm{CY}^{2n_\ell}(Y_L,[2f_\ell\cdot\ell],(j,m_j))\}_{j,m_j}\end{aligned}$$

is in bijection with the decomposition in equivalence classes "j, having multiplicities $m^{(j)}$, of the products, right by left, of the right cycles $\mathrm{CY}^{2n_\ell}(Y_R,[2f_\ell\cdot\ell],(j,m_j))$ shifted in $2f_\ell\cdot\ell$ dimensions by their left equivalents $\mathrm{CY}^{2n_\ell}(Y_L,[2f_\ell\cdot\ell],(j,m_j))$ such that:

$$\begin{aligned}&\mathrm{CY}^{2n_\ell}(Y_R,[2f_\ell\cdot\ell],(j,m_j))\in\mathrm{CH}^{2n_\ell}(Y_R,2f_\ell\cdot\ell)\\ \textit{(resp.}\quad&\mathrm{CY}^{2n_\ell}(Y_L,[2f_\ell\cdot\ell],(j,m_j))\in\mathrm{CH}^{2n_\ell}(Y_L,2f_\ell\cdot\ell)\ )\end{aligned}$$

where $\mathrm{CH}^{2n_\ell}(Y_R,2f_\ell\cdot\ell)$ (resp. $\mathrm{CH}^{2n_\ell}(Y_L,2f_\ell\cdot\ell)$) is the $2n_\ell$-th higher Chow semigroup (see section 2.13).

Proof. This results from section 2.13, the isomorphisms of proposition 4.4 and proposition 4.6. $\blacksquare$

4.6 Proposition

The decomposition of the product, right by left, $\mathrm{CY}^{2n_\ell}(Y_R,[2f_\ell\cdot\ell])\times\mathrm{CY}^{2n_\ell}(Y_L,[2f_\ell\cdot\ell])$ *of cycles of codimension* $2n_\ell$ *shifted in* $2f_\ell{\cdot}\ell$ *dimensions into equivalence class representatives corresponds to the decomposition of* $\mathrm{GL}_{n_\ell[f_\ell\cdot\ell]}((F^T_{\overline{\omega}}\otimes\mathbb{C})\times(F^T_\omega\otimes\mathbb{C}))$, *into the set of products, right by left, of conjugacy class representatives* $g^{2n_\ell}_{T_{R\times L}}([2f_\ell\cdot\ell],(j,m_j))$ *shifted in* $2f_\ell\cdot\ell$ *dimensions:*

$$\begin{aligned}
&\mathrm{CY}^{2n_\ell}(Y_R,[2f_\ell\cdot\ell])\times\mathrm{CY}^{2n_\ell}(Y_L,[2f_\ell\cdot\ell])\\
&\quad=\{(\mathrm{CY}^{2n_\ell}(Y_R,[2f_\ell\cdot\ell],(j,m_j))\times\mathrm{CY}^{2n_\ell}(Y_L,[2f_\ell\cdot\ell],(j,m_j))\}_{j,m_j}\\
&\qquad\qquad\simeq\{\phi(g^{2n_\ell}_{T_R}([2f_\ell\cdot\ell],(j,m_j)))\times\phi(g^{2n_\ell}_{T_L}([2f_\ell\cdot\ell],(j,m_j)))\}_{j,m_j}
\end{aligned}$$

in such a way that each cofunction $\phi(g^{2n_\ell}_{T_R}([2f_\ell\cdot\ell],(j,m_j)))$ *(resp. function* $\phi(g^{2n_\ell}_{T_L}([2f_\ell\cdot\ell],(j,m_j)))$ *) on* $g^{2n_\ell}_{T_R}([2f_\ell\cdot\ell],(j,m_j))$ *(resp.* $g^{2n_\ell}_{T_L}([2f_\ell\cdot\ell],(j,m_j))$ *) be a* n_ℓ*-dimensional complex semitorus* $T^{2n_\ell}_R([2f_\ell\cdot\ell],(j,m_j))$ *(resp.* $T^{2n_\ell}_L([2f_\ell\cdot\ell],(j,m_j))$ *) shifted in* $f_\ell\cdot\ell$ *complex dimensions and localized in the lower (resp. upper) half space (toroidal case only).*

Proof. We have that:

$$\begin{aligned}
&\mathrm{Repsp}(\mathrm{GL}_{n_\ell[f_\ell\cdot\ell]}((F^T_{\overline{\omega}}\otimes\mathbb{C})\times(F^T_\omega\otimes\mathbb{C}))\\
&\qquad\qquad=\{g^{2n_\ell}_{T_R}([2f_\ell\cdot\ell],(j,m_j)))\times g^{2n_\ell}_{T_L}([2f_\ell\cdot\ell],(j,m_j))\}_{j,m_j}\\
&\qquad\qquad\equiv\{g^{2n_\ell}_{T_{R\times L}}([2f_\ell\cdot\ell],(j,m_j))\}_{j,m_j}\ .
\end{aligned}$$

Indeed, $\mathrm{GL}_{n_\ell}(F^T_{\overline{\omega}}\times F^T_\omega)$ decomposes into conjugacy class representatives $g^{(2n_\ell)}_{T_{R\times L}}(j,m_j)\equiv g^{(2n_\ell)}_{T_R}(j,m_j)\times g^{(2n_\ell)}_{T_L}(j,m_j)$ consisting in products, right by left, of n_ℓ-dimensional complex semitori [Pie3].

Then, the bilinear complete semigroup $\mathrm{GL}_{n_\ell[f_\ell\cdot\ell]}((F^T_{\overline{\omega}}\otimes\mathbb{C})\times(F^T_\omega\otimes\mathbb{C})$, shifted in $f_\ell{\cdot}\ell$ complex dimensions from $\mathrm{GL}_{n_\ell}(F^T_{\overline{\omega}}\times F^T_\omega)$, has for conjugacy class representatives $g^{(2n_\ell)}_{T_{R\times L}}([2f_\ell\cdot\ell],(j,m_j))$ which are the conjugacy class representatives $g^{(2n_\ell)}_{T_{R\times L}}(j,m_j)$ of $\mathrm{GL}_{n_\ell}(F^T_{\overline{\omega}}\times F^T_\omega)$ shifted in $2f_\ell\cdot\ell$ dimensions. And, thus, $\phi(g^{2n_\ell}_{T_R}([2f_\ell\cdot\ell],(j,m_j)))\times\phi(g^{2n_\ell}_{T_L}([2f_\ell\cdot\ell],(j,m_j)))$ consists of the product, right by left, of the analytical representatives of n_ℓ-dimensional complex semitori shifted in $f_\ell\cdot\ell$ dimensions. ■

4.7 Proposition

Let $z_{n_\ell}=\sum\limits_{\alpha=1}^{2n_\ell}z_\alpha\ \overrightarrow{e}_\alpha$, $z_{f_\ell\cdot\ell}=\sum\limits_{\beta=1}^{2f_\ell\cdot\ell}z_\beta\ \overrightarrow{e}_\beta$ *and* $(z_{n_\ell-f_\ell\cdot\ell})=\sum\limits_{\gamma=1}^{2(n_\ell-f_\ell\cdot\ell)}z_\gamma\ \overrightarrow{e}_\gamma$ *be respectively a vector of* $\mathbb{C}^{\,n_\ell}$, $\mathbb{C}^{\,f_\ell\cdot\ell}$ *and* $\mathbb{C}^{\,n_\ell-f_\ell\cdot\ell}$.

Then, every left (resp. right) n_ℓ *-dimensional complex semitorus* $T_L^{2n_\ell}([2f_\ell\cdot\ell],(j,m_j))$ *(resp.* $T_R^{2n_\ell}([2f_\ell\cdot\ell],(j,m_j))$ *) shifted in* $f_\ell\cdot\ell$ *complex dimensions and localized in the upper (resp. lower) half space has the following analytic development:*

$$\begin{aligned} T_L^{2n_\ell}([2f_\ell\cdot\ell],(j,m_j)) &\simeq E_{2f_\ell\cdot\ell}(2n_\ell,j,m_j)\ \lambda^{\frac12}(2n_\ell,j,m_j)\ e^{2\pi ijz_{n_\ell}}\\ &\equiv \prod_{c=1}^{2(n_\ell-f_\ell\cdot\ell)} \lambda_c^{\frac12}(2n_\ell,j,m_j)\ e^{2\pi ijz_{n_\ell-f_\ell\cdot\ell}}\\ &\qquad \prod_{d=1}^{2f_\ell\cdot\ell} \lambda_d^{\frac12}(2n_\ell,j,m_j)\ E_d(2n_\ell,j,m_j)\ e^{2\pi ijz_{f_\ell\cdot\ell}}\\ \text{(resp.}\quad T_R^{2n_\ell}([2f_\ell\cdot\ell],(j,m_j)) &\simeq E_{2f_\ell\cdot\ell}(2n_\ell,j,m_j)\ \lambda^{\frac12}(2n_\ell,j,m_j)\ e^{-2\pi ijz_{n_\ell}}\ \text{)}\end{aligned}$$

where:

- $\lambda^{\frac12}(2n_\ell,j,m_j)=\prod\limits_{c=1}^{2(n_\ell-f_\ell\cdot\ell)}\lambda_c^{\frac12}(2n_\ell,j,m_j)\prod\limits_{d=1}^{2f_\ell\cdot\ell}\lambda_d^{\frac12}(2n_\ell,j,m_j)\simeq(j\centerdot N)^{2n_\ell}$ *;*
- $E_{2f_\ell\cdot\ell}(2n_\ell,j,m_j)=\prod\limits_{d=1}^{2f_\ell\cdot\ell}E_d(2n_\ell,j,m_j)$ *is the shift of the Hecke character* $\lambda^{\frac12}(2n_\ell,j,m_j)$ *in such a way that* $E_d(2n_\ell,j,m_j)$ *be a generator of the Lie algebra component* $d_{f_\ell\cdot\ell}(\mathcal{O}_{F_{\omega_{j,m_j}}})\in D_{f_\ell\cdot\ell}(\mathcal{O}_{F_\omega})$ *(see proposition 1.10).*

Proof. 1) According to propositions 4.3, 4.4 and 4.5, the cohomology $H^{2n_\ell-2f_\ell\cdot\ell}(\overline{S}^{P_{n[f_\ell\cdot\ell]}}_{\mathrm{GL}_{n[f_\ell\cdot\ell]}},\widehat{M}^{2n_\ell}_{T_R}[2f_\ell\cdot\ell]\otimes\widehat{M}^{2n_\ell}_{T_L}[2f_\ell\cdot\ell])$, associated with an endomorphism of $\overline{S}^{P_{n_\ell[f_\ell\cdot\ell]}}_{\mathrm{GL}_{n_\ell[f_\ell\cdot\ell]}}$ into itself, decomposes into conjugacy class functional representatives $\phi(g^{2n_\ell}_{T_{R\times L}}([2f_\ell\cdot\ell],(j,m_j)))$ which correspond to the cosets of $\mathrm{GL}_{n_\ell[f_\ell\cdot\ell]}((F^T_{\overline{\omega}}\otimes\mathbb{C})\times(F^T_\omega\otimes\mathbb{C}))\big/\mathrm{GL}_{n_\ell[f_\ell\cdot\ell]}((\mathbb{Z}/N\,\mathbb{Z})^2\otimes\mathbb{C}^2)$. So, the scalar $(E^2_{2f_\ell\cdot\ell}(2n_\ell,j,m_j)\centerdot\lambda(2n_\ell,j,m_j)$ will correspond to the eigenvalues of the (j,m_j)-th coset representative of the Hecke shifted bioperator $T_R(2n_\ell[2f_\ell\cdot\ell];t)\otimes T_L(2n_\ell[2f_\ell\cdot\ell];t)$, since it has a representation into the Lie algebra of $\mathrm{GL}_{n_\ell}((\mathbb{Z}/N\ \mathbb{Z})^2)$ according to section 4.1 and proposition 4.2, while the scalar $\lambda(2n_\ell,j,m_j)$ will correspond to the eigenvalues of the (j,m_j)-th coset representative of the Hecke bioperator $(T_R(2n_\ell;t)\otimes T_L(2n_\ell;t))$ by means of the equality:

$$\lambda(2n_\ell,j,m_j)=\prod_{c=1}^{2(n_\ell-f_\ell\cdot\ell)}\lambda_c(2n_\ell,j,m_j)\prod_{d=1}^{2f_\ell\cdot\ell}\lambda_d(2n_\ell,j,m_j)\ .$$

Remark that the eigenvalues of the (j,m_j)-coset representative of $(T_R(2n_\ell[2f_\ell\cdot\ell];t)\otimes T_L(2n_\ell[2f_\ell\cdot\ell];t)$ are partitioned into unshifted eigenvalues $\lambda_c(2n_\ell,j,m_j)$ and into shifted eigenvalues $(\lambda_d(2n_\ell,j,m_j)\centerdot E^2_d(2n_\ell,j,m_j))$ such that:

$$\begin{aligned}&E^2_{2f_\ell\cdot\ell}(2n_\ell,j,m_j)\centerdot\lambda(2n_\ell,j,m_j)\\ &\qquad=\prod_{c=1}^{2(n_\ell-f_\ell\cdot\ell)}\lambda_c(2n_\ell,j,m_j)\prod_{d=1}^{2f_\ell\cdot\ell}\lambda_d(2n_\ell,j,m_j)\centerdot E^2_d(2n_\ell,j,m_j)\ .\end{aligned}$$

$\lambda(2n_\ell,j,m_j)\simeq j^{2n_\ell}\centerdot N^{2n_\ell}$ since $\lambda(2n_\ell,j,m_j)=\det(\alpha_{2n_\ell^2,j^2}\times D_{j^2,m_j^2})_{\rm ss}$ where D_{j^2,m_j^2} is the decomposition group element of the (j,m_j)-th bisublattice $(\Lambda_{\overline{\omega}_{j,m_j}}\otimes\Lambda_{\omega_{j,m_j}})$ and where $\alpha_{2n_\ell^2,j^2}$ is the corresponding split Cartan subgroup [Pie3].

2) On the other hand, the (j,m_j)-th conjugacy class functional representative $\phi(g^{2n_\ell}_{T_L}([2f_\ell\cdot\ell],(j,m_j)))$ (resp. $\phi(g^{2n_\ell}_{T_R}([2f_\ell\cdot\ell],(j,m_j))))$ is generated by means of the global Frobenius substitution:

$$e^{2\pi iz_{n_\ell}}\longrightarrow e^{2\pi ijz_{n_\ell}}\qquad(\text{resp.}\quad e^{-2\pi iz_{n_\ell}}\longrightarrow e^{-2\pi ijz_{n_\ell}}\ )$$

from the 1-th conjugacy class functional representative

$$\begin{aligned}\phi(g^{2n_\ell}_{T_L}([2f_\ell\cdot\ell],1))&\simeq E_{2f_\ell\cdot\ell}(2n_\ell,1)\ \lambda^{\frac12}(2n_\ell,1)\ e^{2\pi iz_{n_\ell}}\\(\text{resp.}\quad\phi(g^{2n_\ell}_{T_R}([2f_\ell\cdot\ell],1))&\simeq E_{2f_\ell\cdot\ell}(2n_\ell,1)\ \lambda^{\frac12}(2n_\ell,1)\ e^{-2\pi iz_{n_\ell}}\ )\end{aligned}$$

which is a n_ℓ-dimensional complex semitorus shifted in $f_\ell\cdot\ell$ dimensions and localized in the upper (resp. lower) half space. ■

4.8 Proposition

The cohomology $H^{2n_\ell-2f_\ell\cdot\ell}(\overline{S}^{P_{n[f_\ell\cdot\ell]}}_{{\rm GL}_{n[f_\ell\cdot\ell]}},\widehat{M}^{2n_\ell}_{T_R}[2f_\ell\cdot\ell]\otimes\widehat{M}^{2n_\ell}_{T_L}[2f_\ell\cdot\ell])$ *has the analytic development:*

$$\begin{aligned}&H^{2n_\ell-2f_\ell\cdot\ell}(\overline{S}^{P_{n[f_\ell\cdot\ell]}}_{{\rm GL}_{n[f_\ell\cdot\ell]}},\widehat{M}^{2n_\ell}_{T_{R_\oplus}}[2f_\ell\cdot\ell]\otimes\widehat{M}^{2n_\ell}_{T_{L_\oplus}}[2f_\ell\cdot\ell])\\&\quad=\left[\bigoplus_{j=1}^{r}\bigoplus_{m_j}(E_{2f_\ell\cdot\ell}(2n_\ell,j,m_j)\ \lambda^{\frac12}(2n_\ell,j,m_j)\ e^{-2\pi ijz_{n_\ell}}\right]\\&\qquad\qquad\times\left[\bigoplus_{j=1}^{r}\bigoplus_{m_j}(E_{2f_\ell\cdot\ell}(2n_\ell,j,m_j)\ \lambda^{\frac12}(2n_\ell,j,m_j)\ e^{+2\pi ijz_{n_\ell}}\right]\end{aligned}$$

according to the conjugacy class representatives $g^{2n_\ell}_{T_{R\times L}}([2f_\ell\cdot\ell],(j,m_j))$ *where:*

$$\begin{aligned}{\rm EIS}_L(2n_\ell[2f_\ell\cdot\ell],(j,m_j))&=\bigoplus_{j=1}^{r}\bigoplus_{m_j}E_{2f_\ell\cdot\ell}(2n_\ell,j,m_j)\ \lambda^{\frac12}(2n_\ell,j,m_j)\ e^{2\pi ijz_{n_\ell}}\\&\qquad\qquad r\le\infty\ ,\\(\textit{resp.}\quad{\rm EIS}_R(2n_\ell[2f_\ell\cdot\ell],(j,m_j))&=\bigoplus_{j=1}^{r}\bigoplus_{m_j}E_{2f_\ell\cdot\ell}(2n_\ell,j,m_j)\ \lambda^{\frac12}(2n_\ell,j,m_j)\ e^{-2\pi ijz_{n_\ell}}\ )\end{aligned}$$

is the (truncated) Fourier development of a normalized $2n_\ell$*-dimensional left (resp. right) shifted cusp form of weight* $k=2$ *restricted to the upper (resp. lower) half space* [Kub] *(see also* [Pie3]), *chapter3, for the introduction of the equivalent unshifted cusp form).*

Sketch of the proof. This directly results from the decomposition of $H^{2n_\ell-2f_\ell\cdot\ell}(\overline{S}^{P_{n[f_\ell\cdot\ell]}}_{\mathrm{GL}_{n[f_\ell\cdot\ell]}},\widehat{M}^{2n_\ell}_{T_{R_\oplus}}[2f_\ell\cdot\ell]\otimes\widehat{M}^{2n_\ell}_{T_{L_\oplus}}[2f_\ell\cdot\ell])$ into conjugacy class functional representatives $\phi(g^{2n_\ell}_{T_{R\times L}}([2f_\ell\cdot\ell],j,m_j))$ whose analytic representations are given in proposition 4.7. ■

4.9 Theorem (Origin of the (bilinear) spectral theorem)

The analytic development of the cohomology $H^{2n_\ell-2f_\ell\cdot\ell}(\overline{S}^{P_{n[f_\ell\cdot\ell]}}_{\mathrm{GL}_{n[f_\ell\cdot\ell]}},\widehat{M}^{2n_\ell}_{T_{R_\oplus}}[2f_\ell\cdot\ell]\otimes\widehat{M}^{2n_\ell}_{T_{L_\oplus}}[2f_\ell\cdot\ell])$ *gives rise to the eigen(bi)value equation:*

$$\begin{aligned}(D_R^{2f_\ell\cdot\ell}\otimes D_L^{2f_\ell\cdot\ell})&(\mathrm{EIS}_R(2n_\ell,j^{\mathrm{up}}=j,m_j)\otimes(\mathrm{EIS}_L(2n_\ell,j^{\mathrm{up}}=j,m_j))\\ &=E^2_{2f_\ell\cdot\ell}(2n_\ell,j,m_j)(\mathrm{EIS}_R(2n_\ell,j^{\mathrm{up}}=j,m_j)\otimes(\mathrm{EIS}_L(2n_\ell,j^{\mathrm{up}}=j,m_j))\end{aligned}$$

where:

- $(D_R^{2f_\ell\cdot\ell}\otimes D_L^{2f_\ell\cdot\ell})$ *acts on the space of smooth (unshifted) bisections* $\phi(g^{2n_\ell}_{T_{R\times L}}(j,m_j))$ *of* $(\widehat{M}^{2n_\ell}_{T_R}\otimes\widehat{M}^{2n_\ell}_{T_L})$ *such that* $\phi(g^{2n_\ell}_{T_L}(j,m_j))$ *(resp.* $\phi(g^{2n_\ell}_{T_R}(j,m_j))$ *) be a* C^∞*-function localized in the upper (resp. lower) half space (see definition 3.3);*
- ***the eigenvalues*** $\boldsymbol{E^2_{2f_\ell\cdot\ell}(2n_\ell,j,m_j)}$ ***are the shifts of the corresponding generalized Hecke (bi)characters*** $\boldsymbol{\lambda(2n_\ell,j,m_j)}$ ***;***
- ***the eigenbivectors*** $\boldsymbol{\mathrm{EIS}_R(2n_\ell,j^{\mathrm{up}}=j,m_j)\otimes(\mathrm{EIS}_L(2n_\ell,j^{\mathrm{up}}=j,m_j)}$ ***are (tensor) products of truncated Fourier developments at the "*** $\boldsymbol{j}$ ***" classes of normalized*** $\boldsymbol{2n_\ell}$***-dimensional cusp forms,*** $\boldsymbol{j}$ ***varying from*** $\boldsymbol{1}$ ***to*** $\boldsymbol{r}$ ***.***

The set of $\boldsymbol{r}$ ***-tuples:*** $\boldsymbol{\{\mathrm{EIS}_R(2n_\ell,1,m_1)\otimes(\mathrm{EIS}_L(2n_\ell,1,m_1),\dots,\mathrm{EIS}_R(2n_\ell,j^{\mathrm{up}}=j,m_j)\otimes(\mathrm{EIS}_L(2n_\ell,j^{\mathrm{up}}=j,m_j),\dots,\mathrm{EIS}_R(2n_\ell,j^{\mathrm{up}}=r,m_r)\otimes(\mathrm{EIS}_L(2n_\ell,j^{\mathrm{up}}=r,m_r)\}}$ ***is the toroidal spectral representation of the elliptic bioperator*** $\boldsymbol{(D_R^{2f_\ell\cdot\ell}\otimes D_L^{2f_\ell\cdot\ell})\in\mathcal{D}_R\otimes\mathcal{D}_L}$ ***.***
The spectral measure μ_{EIS_L} *(resp.* μ_{EIS_R} *) on the spectrum* $\sigma(D_L^{2f_\ell\cdot\ell})$ *(resp.* $\sigma(D_R^{2f_\ell\cdot\ell})$ *) of* $D_L^{2f_\ell\cdot\ell}$ *(resp.* $D_R^{2f_\ell\cdot\ell}$ *) can be assumed to be the Haar measure.*

Proof. 1) The $\mathrm{GL}_{n_\ell}(F_{\overline{\omega}_\oplus}\times F_{\omega_\oplus})$-bisemimodule $(M^{2n_\ell}_{R_\oplus}\otimes M^{2n_\ell}_{L_\oplus})$ decomposes into subbisemimodules under the endomorphism

$$E_{D_R}\otimes E_{D_L}:\quad M^{2n_\ell}_{R_\oplus}\otimes M^{2n_\ell}_{L_\oplus}\longrightarrow\bigoplus_j\bigoplus_{m_j}(M^{2n_\ell}_{\overline{\omega}_{j,m_j}}\otimes M^{2n_\ell}_{\omega_{j,m_j}})$$

generated under the action of the Hecke bialgebra $\mathcal{H}_{R\times L}(n)$ according to proposition 1.10.

2) There exists a toroidal isomorphism of compactification

$$\gamma_{R\times L}: \qquad M^{2n_\ell}_{R_\oplus}\otimes M^{2n_\ell}_{L_\oplus} \longrightarrow M^{2n_\ell}_{T_{R_\oplus}}\otimes M^{2n_\ell}_{T_{L_\oplus}}$$

sending $(M^{2n_\ell}_{R_\oplus}\otimes M^{2n_\ell}_{L_\oplus})$ into its toroidal equivalent $M^{2n_\ell}_{T_{R_\oplus}}\otimes M^{2n_\ell}_{T_{L_\oplus}}$ according to section 1.12, such that:

$$(M^{2n_\ell}_{T_{R_\oplus}}\otimes M^{2n_\ell}_{T_{L_\oplus}}) = \bigoplus_j \bigoplus_{m_j} (M^{2n_\ell}_{T_{\overline{\omega}_{j,m_j}}}\otimes M^{2n_\ell}_{T_{\omega_{j,m_j}}})$$

has an analytic development given by $\mathrm{EIS}_R(2n_\ell,j,m_j)\otimes \mathrm{EIS}_L(2n_\ell,j,m_j)$ (see proposition 4.8 and [Pie3].

3) • The elliptic bioperator $(D_R^{2f_\ell\cdot\ell}\otimes D_L^{2f_\ell\cdot\ell})$ maps $(\widehat{M}^{2n_\ell}_{R_\oplus}\otimes \widehat{M}^{2n_\ell}_{L_\oplus})$ into its shifted equivalent $\widehat{M}^{2n_\ell}_{R_\oplus}[2f_\ell\cdot\ell]\otimes \widehat{M}^{2n_\ell}_{L_\oplus}[2f_\ell\cdot\ell]$ according to:

$$D_R^{2f_\ell\cdot\ell}\otimes D_L^{2f_\ell\cdot\ell}: \qquad \widehat{M}^{2n_\ell}_{R_\oplus}\otimes \widehat{M}^{2n_\ell}_{L_\oplus} \longrightarrow \widehat{M}^{2n_\ell}_{R_\oplus}[2f_\ell\cdot\ell]\otimes \widehat{M}^{2n_\ell}_{L_\oplus}[2f_\ell\cdot\ell]\ .$$

• $\widehat{M}^{2n_\ell}_{R_\oplus}[2f_\ell\cdot\ell]\otimes \widehat{M}^{2n_\ell}_{L_\oplus}[2f_\ell\cdot\ell]$ is transformed by the unitary action of $(\gamma_{R\times L}\circ(E_{D_R}\otimes E_{D_L}))$ into:

$$\begin{aligned}\gamma_{R\times L}\circ(E_{D_R}\otimes E_{D_L}): \quad & \widehat{M}^{2n_\ell}_{R_\oplus}[2f_\ell\cdot\ell]\otimes \widehat{M}^{2n_\ell}_{L_\oplus}[2f_\ell\cdot\ell]\\ &\longrightarrow \bigoplus_j \bigoplus_{m_j} (\widehat{M}^{2n_\ell}_{T_{\overline{\omega}_{j,m_j}}}[2f_\ell\cdot\ell]\otimes \widehat{M}^{2n_\ell}_{T_{\omega_{j,m_j}}}[2f_\ell\cdot\ell])\\ &\equiv \bigoplus_j \bigoplus_{m_j} \phi(g^{2n_\ell}_{T_{R\times L}}([2f_\ell\cdot\ell],(j,m_j)))\end{aligned}$$

in such a way that the eigenbivalue equation

$$\begin{aligned}&D_R^{2f_\ell\cdot\ell}\otimes D_L^{2f_\ell\cdot\ell}(\mathrm{EIS}_R(2n_\ell,j^{\mathrm{up}}=j,m_j)\otimes \mathrm{EIS}_L(2n_\ell,j^{\mathrm{up}}=j,m_j))\\ &= E^2_{2f_\ell\cdot\ell}(2n_\ell,j,m_j)_{\mathrm{eig}}((\mathrm{EIS}_R(2n_\ell,j^{\mathrm{up}}=j,m_j)\otimes(\mathrm{EIS}_L(2n_\ell,j^{\mathrm{up}}=j,m_j))\end{aligned}$$

corresponds to the map:

$$D_R^{2f_\ell\cdot\ell}\otimes D_L^{2f_\ell\cdot\ell}: \qquad \widehat{M}^{2n_\ell}_{T_{R_\oplus}}\otimes \widehat{M}^{2n_\ell}_{T_{L_\oplus}} \longrightarrow \widehat{M}^{2n_\ell}_{T_{R_\oplus}}[2f_\ell\cdot\ell]\otimes \widehat{M}^{2n_\ell}_{T_{L_\oplus}}[2f_\ell\cdot\ell]\ .$$

4) Indeed, according to proposition 4.6, we have that:

$$M^{2n_\ell}_{T_R}[2f_\ell\cdot\ell]\otimes M^{2n_\ell}_{T_L}[2f_\ell\cdot\ell] \simeq \mathrm{Repsp}(\mathrm{GL}_{n_\ell[f_\ell\cdot\ell]}((F^T_{\overline{\omega}}\otimes\mathbb{C})\times(F^T_\omega\otimes\mathbb{C})))\ .$$

But, $\mathrm{GL}_{n_\ell[f_\ell\cdot\ell]}((F^T_{\overline{\omega}}\otimes\mathbb{C})\times(F^T_\omega\otimes\mathbb{C}))$ is a complete solvable bilinear semigroup implying the chain of embedded normal bilinear subsemigroups:

$$\begin{aligned}&g^{2n_\ell}_{T_{R\times L}}([2f_\ell\cdot\ell],1)\subset\dots\\ &\qquad\subset\dots\subset \bigoplus_{j=1}^{j}\bigoplus_{m_j} g^{2n_\ell}_{T_{R\times L}}([2f_\ell\cdot\ell],(j,m_j))\subset\dots\\ &\qquad\qquad\dots\subset \bigoplus_{j=1}^{r}\bigoplus_{m_j} g^{2n_\ell}_{T_{R\times L}}([2f_\ell\cdot\ell],(j,m_j))\ .\end{aligned}$$

The analytic representation of the j-th normal bilinear subsemigroup: $\overset{j}{\underset{j=1}{\oplus}} \underset{m_j}{\oplus} g^{2n_\ell}_{T_{R\times L}}([2f_\ell\cdot\ell],(j,m_j))$ is precisely the product $\mathrm{EIS}_R(2n_\ell[2f_\ell\cdot\ell],j^{\rm up}=j,m_j)\otimes \mathrm{EIS}_L(2n_\ell[2f_\ell\cdot\ell],j^{\rm up}=j,m_j)$ of Fourier truncated series at "j" classes of normalized shifted $2n_\ell$-dimensional cusp forms.

So, $\mathrm{EIS}_R(2n_\ell[2f_\ell\cdot\ell],j^{\rm up}=j,m_j)\otimes \mathrm{EIS}_L(2n_\ell[2f_\ell\cdot\ell],j^{\rm up}=j,m_j)$ is the j-th analytic representative of $(M^{2n_\ell}_{T_{R_\oplus}}[2f_\ell\cdot\ell]\otimes M^{2n_\ell}_{T_{L_\oplus}}[2f_\ell\cdot\ell])$ and develops as follows:

$$\begin{aligned}
&\mathrm{EIS}_R(2n_\ell[2f_\ell\cdot\ell],j^{\rm up}=j,m_j)\otimes \mathrm{EIS}_L(2n_\ell[2f_\ell\cdot\ell],j^{\rm up}=j,m_j)\\
&\qquad=\left(\overset{j}{\underset{j=1}{\oplus}} \underset{m_j}{\oplus}(E_{2f_\ell\cdot\ell}(2n_\ell,j,m_j)\ \lambda^{\frac12}(2n_\ell,j,m_j)\ e^{-2\pi ijz_{n_\ell}})\right)\\
&\qquad\qquad\otimes\left(\overset{j}{\underset{j=1}{\oplus}} \underset{m_j}{\oplus}(E_{2f_\ell\cdot\ell}(2n_\ell,j,m_j)\ \lambda^{\frac12}(2n_\ell,j,m_j)\ e^{2\pi ijz_{n_\ell}})\right)\\
&\qquad=E^2_{2f_\ell\cdot\ell}(2n_\ell,j,m_j)_{\rm eig}(\mathrm{EIS}_R(2n_\ell,j^{\rm up}=j,m_j)\otimes(\mathrm{EIS}_L(2n_\ell,j^{\rm up}=j,m_j))
\end{aligned}$$

according to proposition 4.8, where:

- $E_{2f_\ell\cdot\ell}(2n_\ell,j,m_j)_{\rm eig}=\overset{j}{\underset{j=1}{\oplus}} \underset{m_j}{\oplus} E_{2f_\ell\cdot\ell}(2n_\ell,j,m_j)$
- $\mathrm{EIS}_L(2n_\ell,j^{\rm up}=j,m_j)=\overset{j}{\underset{j=1}{\oplus}} \underset{m_j}{\oplus}\lambda^{\frac12}(2n_\ell,j,m_j)\ e^{+2\pi ijz_{n_\ell}}$.

5) Thus, we have that:

- $E^2_{2f_\ell\cdot\ell}(2n_\ell,j,m_j)_{\rm eig}$ is the j-th eigenbivalue of the elliptic bioperator $D^{2f_\ell\cdot\ell}_R\otimes D^{2f_\ell\cdot\ell}_L$;
- $(\mathrm{EIS}_R(2n_\ell,j^{\rm up}=j,m_j)\otimes(\mathrm{EIS}_L(2n_\ell,j^{\rm up}=j,m_j))$ is the corresponding j-th eigenbifunction which is also the eigenbifunction of the product of Hecke operators $T_R(2n;j)\otimes T_L(2n;j)$ according to proposition 1.10 and [Pie3].

And the set of r-tuples:

$$\begin{aligned}
\{&\mathrm{EIS}_R(2n_\ell,j^{\rm up}=1,m_1)\otimes \mathrm{EIS}_L(2n_\ell,j^{\rm up}=1,m_1),\dots,\\
&\qquad\dots,\mathrm{EIS}_R(2n_\ell,j^{\rm up}=j,m_j)\otimes \mathrm{EIS}_L(2n_\ell,j^{\rm up}=j,m_j),\dots,\\
&\qquad\qquad\dots,\mathrm{EIS}_R(2n_\ell,j^{\rm up}=r,m_r)\otimes \mathrm{EIS}_L(2n_\ell,j^{\rm up}=r,m_r)\}
\end{aligned}$$

is the toroidal spectral representation of $D^{2f_\ell\cdot\ell}_R\otimes D^{2f_\ell\cdot\ell}_L$. ■

4.10 Corollary

Let $\widehat{M}^{2n_\ell}_{T_{R_\oplus}} \otimes \widehat{M}^{2n_\ell}_{T_{L_\oplus}}$ *be the (truncated) normalized cusp biform over the* $\mathrm{GL}_{n_\ell}(F_{\overline{\omega}_\oplus} \otimes F_{\omega_\oplus})$ *-bisemimodule*

$$M^{2n_\ell}_{R_\oplus} \otimes M^{2n_\ell}_{L_\oplus} = \bigoplus_{j=1}^{j} \bigoplus_{m_j} (M^{2n_\ell}_{\overline{\omega}_{j,m_j}} \otimes M^{2n_\ell}_{\omega_{j,m_j}})$$

decomposing into the sum of subbisemimodules $(M^{2n_\ell}_{\overline{\omega}_{j,m_j}} \otimes M^{2n_\ell}_{\omega_{j,m_j}})$ *according to the conjugacy classes of the complete bilinear semigroup* $\mathrm{GL}_{n_\ell}(F_{\overline{\omega}} \times F_\omega)$ *having multiplicities* $m^{(j)} = \sup(m_j)$ *,* m_j *being an increasing integer superior or equal to* 1 *.*
Then, there exist Haar bimeasures $\mu_{j_R} \times \mu_{j_L}$ *on the spectrum* $\sigma(D_R^{2f_\ell\cdot\ell} \otimes D_L^{2f_\ell\cdot\ell})$ *of the elliptic bioperator* $(D_R^{2f_\ell\cdot\ell} \otimes D_L^{2f_\ell\cdot\ell})$ *and an isomorphism*

$$\begin{aligned}\gamma_{R\times L} \circ (E_{D_R} \otimes E_{D_L}) : \qquad & \widehat{M}^{2n_\ell}_{R_\oplus} \otimes \widehat{M}^{2n_\ell}_{L_\oplus} \\ & \longrightarrow \bigoplus_{j=1}^{j} \bigoplus_{m_j} ((\mathrm{EIS}_R(2n_\ell, j, m_j) \otimes \mathrm{EIS}_L(2n_\ell, j, m_j))\end{aligned}$$

leading to the eigenbivalue equation(s):

$$[(\gamma_R \circ D_R^{2f_\ell\cdot\ell}) \otimes (D_L^{2f_\ell\cdot\ell} \circ \gamma_L)(\widehat{M}^{2n_\ell}_{R_\oplus} \otimes \widehat{M}^{2n_\ell}_{L_\oplus}] = E^2_{2f_\ell\cdot\ell}(2n_\ell, j, m_j)_{\mathrm{eig}}(\widehat{M}^{2n_\ell}_{R_\oplus} \otimes \widehat{M}^{2n_\ell}_{L_\oplus})$$

whose spectral representation is given by the set of eigenbifunctions $\{\mathrm{EIS}_R(2n_\ell, j^{\mathrm{up}} = j, m_j) \otimes \mathrm{EIS}_L(2n_\ell, j^{\mathrm{up}} = j, m_j)\}^r_{j=1,m_j}$ *having multiplicities* $m^{(j)} = \sup(m_j)$ *.*

4.11 Proposition

A trace formula [Art] ***corresponding to a shifted Plancherel formula*** and *associated with the* j *-th eigenbifunction* $\mathrm{EIS}_R(2n_\ell, j^{\mathrm{up}} = j, m_j) \otimes \mathrm{EIS}_L(2n_\ell, j^{\mathrm{up}} = j, m_j)$ *of the eigenvalue equation:*

$$[(\gamma_R \circ D_R^{2f_\ell\cdot\ell}) \otimes (D_L^{2f_\ell\cdot\ell} \circ \gamma_L)(\widehat{M}^{2n_\ell}_{R_\oplus} \otimes \widehat{M}^{2n_\ell}_{L_\oplus}] = E^2_{2f_\ell\cdot\ell}(2n_\ell, j, m_j)_{\mathrm{eig}}(\widehat{M}^{2n_\ell}_{R_\oplus} \otimes \widehat{M}^{2n_\ell}_{L_\oplus})$$

is given by the ***bilinear form****:*

$$\begin{aligned}&(\mathrm{EIS}_R(2n_\ell[2f_\ell\cdot\ell], j^{\mathrm{up}} = j, m_j), \mathrm{EIS}_L(2n_\ell[2f_\ell\cdot\ell], j^{\mathrm{up}} = j, m_j)) \\ &\qquad = \bigoplus_{j=1}^{j} \bigoplus_{m_j} (\lambda(2n_\ell, j, m_j)\ E^2_{2f_\ell\cdot\ell}(2n_\ell, j, m_j))\end{aligned}$$

from $\mathrm{FRepsp}(\mathrm{GL}_{n_\ell[f_\ell\cdot\ell]}((F^T_{\overline{\omega}_\oplus} \otimes \mathbb{C}) \times (F^T_{\omega_\oplus} \otimes \mathbb{C})))$ *to* $\mathbb{C}$ *.*

Proof. This trace formula directly results from point 4) of the proof of proposition 4.9 and corresponds to the shifted Plancherel formula since the trace formula

$$(\mathrm{EIS}_R(2n_\ell, j, m_j), \mathrm{EIS}_L(2n_\ell, j, m_j)) = \bigoplus_{j=1}^{r} \lambda(2n_\ell, j, m_j)$$

from $\mathrm{FRepsp}(\mathrm{GL}_{n_\ell}(F^T_{\overline{\omega}_\oplus} \otimes F^T_{\omega_\oplus}))$ to $\mathbb{C}$ is the **Plancherel formula** associated with the bilinear semigroup $\mathrm{GL}_{n_\ell}(F^T_{\overline{\omega}} \times F^T_{\omega})$. ■

4.12 Proposition

The product, right by left, $\mathrm{EIS}_R(2n_\ell[2f_\ell\cdot\ell],j,m_j)\otimes\mathrm{EIS}_L(2n_\ell[2f_\ell\cdot\ell],j,m_j)$ *of the truncated Fourier development of the shifted* $2n_\ell$ *dimensional cusp biform, constitutes:*

1) *a supercuspidal representation of the shifted complete bilinear semigroup* $\mathrm{GL}_{n_\ell[f_\ell\cdot\ell]}((F^T_{\overline{\omega}}\otimes\mathbb{C})\times(F^T_{\omega}\otimes\mathbb{C}))$ *;*
2) *a shifted supercuspidal representation of the complete bilinear semigroup* $\mathrm{GL}_{n_\ell}((F^T_{\overline{\omega}}\times F^T_{\omega})$ *.*

Proof. 1) According to proposition 4.8, $\mathrm{EIS}_R(2n_\ell[2f_\ell\cdot\ell],j,m_j)$ (resp. $\mathrm{EIS}_L(2n_\ell[2f_\ell\cdot\ell],j,m_j)$) is the truncated Fourier development of a normalized $2n_\ell$-dimensional right (resp. left) shifted cusp form of weight $k=2$. Consequently, $\mathrm{EIS}_{R\times L}(2n_\ell[2f_\ell\cdot\ell],j,m_j)=\mathrm{EIS}_R(2n_\ell[2f_\ell\cdot\ell],j,m_j)\otimes\mathrm{EIS}_L(2n_\ell[2f_\ell\cdot\ell],j,m_j)$ is a truncated cuspidal biform over $\mathbb{C}^{n_\ell}\times_D\mathbb{C}^{n_\ell}$. On the other hand, as we have the equality:

$$\mathrm{FRepsp}(\mathrm{GL}_{n_\ell[f_\ell\cdot\ell]}((F^T_{\overline{\omega}_\oplus}\otimes\mathbb{C})\times(F^T_{\omega_\oplus}\otimes\mathbb{C})))=\mathrm{EIS}_{R\times L}(2n_\ell[2f_\ell\cdot\ell],j,m_j)$$

according to propositions 4.6 and 4.7, and as $\mathrm{FRepsp}(\mathrm{GL}_{n_\ell[f_\ell\cdot\ell]}((F^T_{\overline{\omega}}\otimes\mathbb{C})\times(F^T_{\omega}\otimes\mathbb{C})))$ is irreducible and the coefficients of $\mathrm{EIS}_{R\times L}(2n_\ell[2f_\ell\cdot\ell],j,m_j)$ have compact support in $\mathrm{GL}_{n_\ell[f_\ell\cdot\ell]}((F^T_{\overline{\omega}}\otimes\mathbb{C})\times(F^T_{\omega}\otimes\mathbb{C}))$, $\mathrm{EIS}_{R\times L}(2n_\ell[2f_\ell\cdot\ell],j,m_j)$ constitutes a supercuspidal representation of $\mathrm{GL}_{n_\ell[f_\ell\cdot\ell]}((F^T_{\overline{\omega}}\otimes\mathbb{C})\times(F^T_{\omega}\otimes\mathbb{C}))$.

2) Taking into account proposition 4.9, it clearly appears that $\mathrm{EIS}_{R\times L}(2n_\ell[2f_\ell\cdot\ell],j,m_j)$ also constitutes a shifted supercuspidal representation of $\mathrm{GL}_{n_\ell}(F^T_{\overline{\omega}}\times F^T_{\omega})$. ■

4.13 Holomorphic spectral representation

This chapter has been essentially devoted to the toroidal spectral representation of an elliptic bioperator given by a set of r-tuples of products, right by left, of truncated Fourier developments of cusp forms.

Indeed, the aim of this paper, put in concrete form in chapter 5, deals with supercuspidal representations of shifted algebraic bilinear semigroups in the frame of geometric-shifted global bilinear correspondences of Langlands.

A cusp form, being a holomorphic function, the conclusions obtained for the toroidal spectral representation of an elliptic bioperator result in fact from its **"holomorphic" spectral representation** as developed succinctly in the next sections of this chapter.
As the toroidal spectral representation of an elliptic bioperator $(D_R^{2f_\ell\cdot\ell}\otimes D_L^{2f_\ell\cdot\ell})$ is directly connected to the functional representation space $\mathrm{FRepsp}(\mathrm{GL}_{n_\ell[f_\ell\cdot\ell]}((F_{\overline{\omega}}^T\otimes\mathbb{C}\,)\times(F_\omega^T\otimes\mathbb{C}\,)))$ of the shifted bilinear complete semigroup $\mathrm{GL}_{n_\ell[f_\ell\cdot\ell]}((F_{\overline{\omega}}^T\otimes\mathbb{C}\,)\times(F_\omega^T\otimes\mathbb{C}\,))$, the associated holomorphic spectral representation will be proved to result from the functional representation space $\mathrm{FRepsp}(\mathrm{GL}_{n_\ell[f_\ell\cdot\ell]}((F_{\overline{\omega}}\otimes\mathbb{C}\,)\times(F_\omega\otimes\mathbb{C}\,)))$ of the shifted bilinear complete semigroup $\mathrm{GL}_{n_\ell[f_\ell\cdot\ell]}((F_{\overline{\omega}}\otimes\mathbb{C}\,)\times(F_\omega\otimes\mathbb{C}\,)))$.

4.14 Proposition

The differential bioperator $(D_R^{2f_\ell\cdot\ell}\otimes D_L^{2f_\ell\cdot\ell})\in\mathcal{D}_R\otimes\mathcal{D}_L$ *maps the bisemisheaf* $(\widehat{M}_R^{2n_\ell}\otimes\widehat{M}_L^{2n_\ell})$ *on the* $\mathrm{GL}_{n_\ell}(F_{\overline{\omega}}\times F_\omega)$*-bisemimodule* $(M_R^{2n_\ell}\otimes M_L^{2n_\ell})$ *into the perverse bisemisheaf* $(\widehat{M}_R^{2n_\ell}[2f_\ell\cdot\ell]\otimes\widehat{M}_L^{2n_\ell}[2f_\ell\cdot\ell])$ *according to:*

$$(D_R^{2f_\ell\cdot\ell}\otimes D_L^{2f_\ell\cdot\ell}):\qquad(\widehat{M}_R^{2n_\ell}\otimes\widehat{M}_L^{2n_\ell})\longrightarrow(\widehat{M}_R^{2n_\ell}[2f_\ell\cdot\ell]\otimes\widehat{M}_L^{2n_\ell}[2f_\ell\cdot\ell])\ .$$

Proof. This is an adaptation of proposition 4.3. ■

4.15 Proposition

The bilinear cohomology $H^{2n_\ell-2f_\ell\cdot\ell}(M_{DM_{R\times L}})(X_{R\times L}^{\mathrm{sv}}),X_R^{2n_\ell}[2f_\ell\cdot\ell]\times X_L^{2n_\ell}[2f_\ell\cdot\ell])$ *of the Suslin-Voevodsky mixed bimotive* $M_{DM_{R\times L}}(X_{R\times L}^{\mathrm{sv}})$ *is isomorphic to the decomposition in conjugacy classes of the product, right by left,* $\mathrm{CY}^{2n_\ell}(Y_R,[2f_\ell\cdot\ell])\times\mathrm{CY}^{2n_\ell}(Y_L,[2f_\ell\cdot\ell])$ *of* $2n_\ell$*-dimensional cycles shifted in* $2f_\ell\cdot\ell$*-dimensions:*

$$\begin{aligned}H^{2n_\ell-2f_\ell\cdot\ell}&(M_{DM_{R\times L}})(X_{R\times L}^{\mathrm{sv}}),X_R^{2n_\ell}[2f_\ell\cdot\ell]\times X_L^{2n_\ell}[2f_\ell\cdot\ell])\\&\simeq\mathrm{CY}^{2n_\ell}(Y_R,[2f_\ell\cdot\ell])\times\mathrm{CY}^{2n_\ell}(Y_L,[2f_\ell\cdot\ell])\\&=\{\mathrm{CY}^{2n_\ell}(Y_R,[2f_\ell\cdot\ell],(j,m_j))\times\mathrm{CY}^{2n_\ell}(Y_L,[2f_\ell\cdot\ell],(j,m_j))\}_{j,m_j}\ .\end{aligned}$$

Proof. This results from the isomorphism

$$\begin{aligned}H^{2n_\ell-2f_\ell\cdot\ell}&(M_{DM_{R\times L}})(X_{R\times L}^{\mathrm{sv}}),X_R^{2n_\ell}[2f_\ell\cdot\ell]\times X_L^{2n_\ell}[2f_\ell\cdot\ell])\\&\simeq H^{2n_\ell-2f_\ell\cdot\ell}(\widehat{M}_R^{2n}[2f_\ell\cdot\ell]\otimes\widehat{M}_L^{2n}[2f_\ell\cdot\ell]),(\widehat{M}_R^{2n_\ell}[2f_\ell\cdot\ell]\otimes\widehat{M}_L^{2n_\ell}[2f_\ell\cdot\ell])\end{aligned}$$

between cohomologies according to propositions 4.4 and 4.5 in such a way that

a) $\widehat{M}_R^{2n_\ell}[2f_\ell \cdot \ell] \otimes \widehat{M}_L^{2n_\ell}[2f_\ell \cdot \ell])$ is the bisemisheaf over the representation space $\mathrm{Repsp}(\mathrm{GL}_{n_\ell[f_\ell\cdot\ell]}((F_{\overline{\omega}} \otimes \mathbb{C}) \times (F_\omega \otimes \mathbb{C})))$ of the bilinear complete semigroup $\mathrm{GL}_{n_\ell[f_\ell\cdot\ell]}((F_{\overline{\omega}} \otimes \mathbb{C}) \times (F_\omega \otimes \mathbb{C}))$;

b) $\mathrm{CY}^{2n_\ell}(Y_R,[2f_\ell \cdot \ell]) \times \mathrm{CY}^{2n_\ell}(Y_L,[2f_\ell \cdot \ell])$ is the $2n_\ell$-th bicycle isomorphic to $\mathrm{GL}_{n_\ell[f_\ell\cdot\ell]}((F_{\overline{\omega}} \otimes \mathbb{C}) \times (F_\omega \otimes \mathbb{C}))$. ■

4.16 Laurent polynomials on $\mathrm{GL}_{n_\ell}(F_{\overline{\omega}} \times F_\omega)$

- Let $g_L^{2n_\ell}(j,m_j)$ (resp. $g_R^{2n_\ell}(j,m_j)$) be the (j,m_j)-th left (resp. right) linear conjugacy class representative of $\mathrm{GL}_{n_\ell}(F_{\overline{\omega}} \times F_\omega)$ and let $\psi(g_L^{2n_\ell}(j,m_j))$ (resp. $\psi(g_R^{2n_\ell}(j,m_j))$) be a differentiable function on it, into $\mathbb{C}$, given simply by

$$\begin{aligned}
\psi(g_L^{2n_\ell}(j,m_j)) &= \lambda^{\frac{1}{2}}(2n_\ell,j,m_j)(y_1^j \times \cdots \times y_{n_\ell}^j)\ , \\
&= \lambda^{\frac{1}{2}}(2n_\ell,j,m_j)y^j\ , \qquad\qquad y = y_1 \times \cdots \times y_{n_\ell}\ , \\
\text{(resp.}\quad \psi(g_R^{2n_\ell}(j,m_j)) &= \lambda^{\frac{1}{2}}(2n_\ell,j,m_j)(y_1^{*j} \times \cdots \times y_{n_\ell}^{*j})\ , \quad y_{n_\ell}^* \text{ being the} \\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{conjugate complex of } y_{n_\ell} \\
&= \lambda^{\frac{1}{2}}(2n_\ell,j,m_j)(y^*)^j\ ,\ )
\end{aligned}$$

where $y_1,\dots,y_{n_\ell}$ are functions of complex variables on unitary closed supports.

- If the conjugacy class representatives of $T_{n_\ell}(F_\omega)$ (resp. $T_{n_\ell}^t(F_{\overline{\omega}})$) $\subset \mathrm{GL}_{n_\ell}(F_{\overline{\omega}} \times F_\omega)$ are glued together, **a Laurent polynomial on the representation space of $\boldsymbol{T_{n_\ell}(F_{\omega_\oplus})}$ (resp. $\boldsymbol{T^t_{n_\ell}(F_{\overline{\omega}_\oplus})}$)** will be defined by:

$$\begin{aligned}
\psi(\mathrm{Repsp}(T_{n_\ell}(F_{\omega_\oplus})) &= \sum_{j=1}^{r}\sum_{m_j} \lambda^{\frac{1}{2}}(2n_\ell,j,m_j)\ y^j\ , \qquad\qquad r \leq \infty \\
\text{(resp.}\quad \psi(\mathrm{Repsp}(T^t_{n_\ell}(F_{\overline{\omega}_\oplus})) &= \sum_{j=1}^{r}\sum_{m_j} \lambda^{\frac{1}{2}}(2n_\ell,j,m_j)\ (y^*)^j\ ,\ )
\end{aligned}$$

where $\lambda^{\frac{1}{2}}(2n_\ell,j,m_j)$ is the square root of the product of the eigenvalues of the (j,m_j)-th coset representative of the Hecke bioperator as described in proposition 4.7.

- And, a Laurent bipolynomial on the representation space $\mathrm{GL}_{n_\ell}(F_{\overline{\omega}_\oplus} \times F_{\omega_\oplus})$ with respect to its conjugacy classes glued together will be given by:

$$\psi(\mathrm{Repsp}(\mathrm{GL}_{n_\ell}(F_{\overline{\omega}_\oplus} \times F_{\omega_\oplus}))) = \sum_{j=1}^{j}\sum_{m_j}(\lambda^{\frac{1}{2}}(2n_\ell,j,m_j) \times (y^*)^j) \times (\lambda^{\frac{1}{2}}(2n_\ell,j,m_j) \times y^j)\ .$$

4.17 Proposition

On the glued together conjugacy class representatives of $\mathrm{GL}_{n_\ell}(F_{\overline{\omega}} \times F_\omega)$ *, the function* $\psi(\mathrm{Repsp}(T_{n_\ell}(F_{\omega_\oplus})))$ *(resp.* $\psi(\mathrm{Repsp}(T^t_{n_\ell}(F_{\overline{\omega}_\oplus})))$ *), defined in a neighbourhood of a point* y'_0 *(resp.* $y^{*\prime}_0$ *) of* $\mathbb{C}^{n_\ell}$ *, is holomorphic at* y'_0 *(resp.* $y^{*\prime}_0$ *) if we have the following multiple power series developments:*

$$\psi(\mathrm{Repsp}(T_{n_\ell}(F_{\omega_\oplus})) = \sum_{j=1}^{r} \sum_{m_j} \lambda'^{\frac{1}{2}}\ (2n_\ell, j, m_j)\ (y' - y'_0)^j$$

$$\textit{(resp.}\quad \psi(\mathrm{Repsp}(T^t_{n_\ell}(F_{\overline{\omega}_\oplus})) = \sum_{j=1}^{r} \sum_{m_j} \lambda'^{\frac{1}{2}}\ (2n_\ell, j, m_j)\ (y^{*'} - y^{*'}_0)^j\ ).$$

And, the holomorphic bifunction

$$\psi(\mathrm{Repsp}(T^t_{n_\ell}(F_{\overline{\omega}_\oplus})) \otimes \psi(\mathrm{Repsp}(T_{n_\ell}(F_{\omega_\oplus})) = \sum_{j=1}^{r} \sum_{m_j} \lambda'\ (2n_\ell, j, m_j)\ (y^{*'} y' - y^{*'}_0 y'_0)^j$$

at the bipoint $(y^{*\prime}_0 y'_0)$ *constitutes an irreducible holomorphic representation* $\mathrm{Irr\,hol}((\mathrm{GL}_{n_\ell}(F_{\overline{\omega}} \times F_\omega)))$ *of the bilinear semigroup* $\mathrm{GL}_{n_\ell}(F_{\overline{\omega}} \times F_\omega)$ *.*

Sketch of proof. This is a consequence of the introduction of Laurent polynomials on $\mathrm{GL}_{n_\ell}(F_{\overline{\omega}} \times F_\omega)$ in section 4.16 and [Pie3]. ■

4.18 Shifted holomorphic representation of $\mathrm{GL}_{n_{\ell[f_\ell \cdot \ell]}}((F_{\overline{\omega}} \otimes \mathbb{C}) \times (F_\omega \otimes \mathbb{C}))$

- Similarly as it was done in proposition 4.7, a function $\psi(\mathrm{Repsp}(T_{n_{\ell[f_\ell \cdot \ell]}}(F_{\omega_\oplus} \otimes \mathbb{C})))$ (resp. a cofunction $\psi(\mathrm{Repsp}(T^t_{n_{\ell[f_\ell \cdot \ell]}}(F_{\overline{\omega}_\oplus} \otimes \mathbb{C})))$) on the representation space of the linear complete semigroup $T_{n_{\ell[f_\ell \cdot \ell]}}(F_{\omega_\oplus} \otimes \mathbb{C}))$ (resp. $T^t_{n_{\ell[f_\ell \cdot \ell]}}(F_{\overline{\omega}_\oplus} \otimes \mathbb{C}))$), shifted in $2f_\ell \cdot \ell$ dimensions, will be introduced by:

$$\begin{aligned} &\psi(\mathrm{Repsp}(T_{n_{\ell[f_\ell \cdot \ell]}}(F_{\omega_\oplus} \otimes \mathbb{C})))) \\ &\qquad = \sum_{j=1}^{r} \sum_{m_j} c_{2f_\ell \cdot \ell}\ (2n_\ell, j, m_j)\ \lambda^{\frac{1}{2}}(2n_\ell, j, m_j)\ (y^j) \\ \text{(resp.}\quad &\psi(\mathrm{Repsp}(T^t_{n_{\ell[f_\ell \cdot \ell]}}(F_{\overline{\omega}_\oplus} \otimes \mathbb{C})))) \\ &\qquad = \sum_{j=1}^{r} \sum_{m_j} c_{2f_\ell \cdot \ell}\ (2n_\ell, j, m_j)\ \lambda^{\frac{1}{2}}(2n_\ell, j, m_j)\ (y^{*j})\ ) \end{aligned}$$

  where: $c_{2f_\ell \cdot \ell}\,(2n_\ell, j, m_j)$ is the shift in $f_\ell \cdot \ell$ complex dimensions of the Hecke character $\lambda^{\frac{1}{2}}(2n_\ell, j, m_j)$.

- And the bifunction

$$\begin{aligned}\psi(\mathrm{Repsp}(T^t_{n_\ell[f_\ell\cdot\ell]}(F_{\overline{\omega}_\oplus}\otimes\mathbb{C}\,)))\otimes\psi(\mathrm{Repsp}(T_{n_\ell[f_\ell\cdot\ell]}(F_{\omega_\oplus}\otimes\mathbb{C}\,)))\\ =\sum_{j=1}^{r}\sum_{m_j} c^2_{2f_\ell\cdot\ell}\ (2n_\ell,j,m_j)\ \lambda(2n_\ell,j,m_j)\ (y^*y)^j\end{aligned}$$

on the representation space of the bilinear complete semigroup $\mathrm{GL}_{n_\ell[f_\ell\cdot\ell]}(F_{\overline{\omega}_\oplus}\otimes\mathbb{C}\,)\times(F_{\omega_\oplus}\otimes\mathbb{C}\,))$, shifted in $f_\ell\cdot\ell$ complex dimensions on its right and left parts, constitutes an irreducible shifted holomorphic representation $\mathrm{Irr\,hol}(\mathrm{GL}_{n_\ell[f_\ell\cdot\ell]}(F_{\overline{\omega}}\otimes\mathbb{C}\,)\times(F_\omega\otimes\mathbb{C}\,)))$ of $\mathrm{GL}_{n_\ell}(F_{\overline{\omega}}\otimes\mathbb{C}\,)\times(F_\omega\otimes\mathbb{C}\,))$.

4.19 Theorem (Holomorphic spectral theorem)

The analytic development of the cohomology

$$\begin{aligned}H^{2n_\ell-2f_\ell\cdot\ell}(\widehat{M}^{2n}_R[2f_\ell\cdot\ell]\otimes\widehat{M}^{2n}_L[2f_\ell\cdot\ell],\widehat{M}^{2n_\ell}_R[2f_\ell\cdot\ell]\otimes\widehat{M}^{2n_\ell}_L[2f_\ell\cdot\ell])\\ =\sum_{j=1}^{r}\sum_{m_j}(c^2_{2f_\ell\cdot\ell}(2n_\ell,j,m_j)\ \lambda(2n_\ell,j,m_j)\ (y^*y)^j\end{aligned}$$

gives rise to the eigen(bi)value equation:

$$\begin{aligned}&(D^{2f_\ell\cdot\ell}_R\otimes D^{2f_\ell\cdot\ell}_L)(\psi(\mathrm{Repsp}(T^t_{n_\ell}(F_{\overline{\omega}_\oplus},j^{\mathrm{up}}=j))\otimes\psi(\mathrm{Repsp}(T_{n_\ell}(F_{\omega_\oplus},j^{\mathrm{up}}=j)))\\ &\quad=c_{2f_\ell\cdot\ell}(2n_\ell,j,m_j)_{\mathrm{eig}}\ (\psi(\mathrm{Repsp}(T^t_{n_\ell}(F_{\overline{\omega}_\oplus},j^{\mathrm{up}}=j))\otimes\psi(\mathrm{Repsp}(T_{n_\ell}(F_{\omega_\oplus},j^{\mathrm{up}}=j)))\end{aligned}$$

where:

- $(D^{2f_\ell\cdot\ell}_R\otimes D^{2f_\ell\cdot\ell}_L)$ *acts on the space of smooth (unshifted) bisections* $\psi(g^{2n_\ell}_{R\times L}(j,m_j))$ *of* $(\widehat{M}^{2n_\ell}_R\otimes\widehat{M}^{2n_\ell}_L)$ *;*
- ***the eigenbivectors*** $(\psi(\mathbf{Repsp}(\boldsymbol{T^t_{n_\ell}}(\boldsymbol{F_{\overline{\omega}_\oplus}},\boldsymbol{j}^{\mathbf{up}}=\boldsymbol{j}))\otimes\psi(\mathbf{Repsp}(\boldsymbol{T_{n_\ell}}(\boldsymbol{F_{\omega_\oplus}},\boldsymbol{j}^{\mathbf{up}}=\boldsymbol{j})))$ ***are tensor products of truncated holomorphic functions at*** $\boldsymbol{j}^{\mathbf{up}}=\boldsymbol{j}$ ***terms in such a way that the*** $\boldsymbol{r}$ ***-tuple:***

$$\begin{aligned}&\{(\psi(\mathrm{Repsp}(T^t_{n_\ell}(F_{\overline{\omega}_\oplus},j^{\mathrm{up}}=1))\otimes\psi(\mathrm{Repsp}(T_{n_\ell}(F_{\omega_\oplus},j^{\mathrm{up}}=1)))\\ &\qquad,\dots,\\ &\qquad(\psi(\mathrm{Repsp}(T^t_{n_\ell}(F_{\overline{\omega}_\oplus},j^{\mathrm{up}}=j))\otimes\psi(\mathrm{Repsp}(T_{n_\ell}(F_{\omega_\oplus},j^{\mathrm{up}}=j)))\\ &\qquad,\dots,\\ &\qquad\qquad(\psi(\mathrm{Repsp}(T^t_{n_\ell}(F_{\overline{\omega}_\oplus},j^{\mathrm{up}}=r))\otimes\psi(\mathrm{Repsp}(T_{n_\ell}(F_{\omega_\oplus},j^{\mathrm{up}}=r)))\}\end{aligned}$$

constitutes the holomorphic spectral representation of the elliptic bioperator $(\boldsymbol{D^{2f_\ell\cdot\ell}_R}\otimes\boldsymbol{D^{2f_\ell\cdot\ell}_L})$ ***;***

- ***the eigenbivalues*** $\boldsymbol{c_{2f_\ell\cdot\ell}(2n_\ell, j, m_j)}$ ***are shifts of the generalized Hecke (bi)characters in the sense of section 4.18.***

Sketch of proof. This theorem is an adaptation to the holomorphic case of the spectral theorem 4.9 having led to a toroidal spectral representation of the elliptic bioperator $(D_R^{2f_\ell\cdot\ell}\otimes D_L^{2f_\ell\cdot\ell})$. ■

4.20 Proposition

1) *The holomorphic spectral representation of theorem 4.19 is isomorphic to the toroidal spectral representation of theorem 4.9.*
2) *Every spectral representation on the representation space of the bilinear complete semigroup* $\mathrm{GL}_{n_\ell}(F_{\overline{\omega}}\times F_\omega)$ *is isomorphic to the holomorphic and toroidal spectral representations mentioned above.*

Proof. Indeed, every spectral representation on a functional representation space $F(\mathrm{Repsp}(\mathrm{GL}_{n_\ell}(F_{\overline{\omega}}\times F_\omega))$ of the bilinear complete semigroup $\mathrm{GL}_{n_\ell}(F_{\overline{\omega}}\times F_\omega)$ has the structure of a r-tuple:

$$\begin{aligned}\big\{&(F(\mathrm{Repsp}(T^t_{n_\ell}(F_{\overline{\omega}_\oplus}, j^{\mathrm{up}}=1))\otimes F(\mathrm{Repsp}(T_{n_\ell}(F_{\omega_\oplus}, j^{\mathrm{up}}=1)))\\ &\qquad ,\dots,\\ &\qquad (F(\mathrm{Repsp}(T^t_{n_\ell}(F_{\overline{\omega}_\oplus}, j^{\mathrm{up}}=j))\otimes F(\mathrm{Repsp}(T_{n_\ell}(F_{\omega_\oplus}, j^{\mathrm{up}}=j)))\\ &\qquad ,\dots,\\ &\qquad (F(\mathrm{Repsp}(T^t_{n_\ell}(F_{\overline{\omega}_\oplus}, j^{\mathrm{up}}=r))\otimes F(\mathrm{Repsp}(T_{n_\ell}(F_{\omega_\oplus}, j^{\mathrm{up}}=r)))\big\}\\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad 1\le j\le r\le\infty\end{aligned}$$

as indicated for the holomorphic and toroidal spectral representations of the elliptic bioperator $(D_R^{2f_\ell\cdot\ell}\otimes D_L^{2f_\ell\cdot\ell})$. ■

5 Geometric-shifted global bilinear correspondences of Langlands

5.1 Lemma

Let $W^{ab}_{F^{S_\mathbb{C}}_{\overline{\omega}}} \times W^{ab}_{F^{S_\mathbb{C}}_{\omega}}$ *be the product of the shifted global Weil (semi)group* $W^{ab}_{F^{S_\mathbb{C}}_{\overline{\omega}}}$ *by its equivalent* $W^{ab}_{F^{S_\mathbb{C}}_{\omega}}$ *as introduced in section 1.5.*
Then, there exists an irreducible representation:

$$\mathrm{Irr}\, W^{(2n_\ell)}_{F_{R\times L}} : \qquad W^{ab}_{F^{S_\mathbb{C}}_{\overline{\omega}}} \times W^{ab}_{F^{S_\mathbb{C}}_{\omega}} \longrightarrow \mathrm{GL}_{n_\ell[f_\ell\cdot\ell]}((F_{\overline{\omega}}\otimes\mathbb{C}\,)\times(F_{\omega^s}\otimes\mathbb{C}\,))$$

from $(W^{ab}_{F^{S_\mathbb{C}}_{\overline{\omega}}} \times W^{ab}_{F^{S_\mathbb{C}}_{\omega}})$ *to the complex bilinear complete semigroup* $\mathrm{GL}_{n_\ell[f_\ell\cdot\ell]}((F_{\overline{\omega}}\otimes\mathbb{C}\,)\times(F_\omega\otimes\mathbb{C}\,))$ *shifted in* $f_\ell\cdot\ell$ *complex dimensions in such a way that* [Pie1], [Pie2]:

1) $G^{(2n_\ell[2f_\ell\cdot\ell])}((F_{\overline{\omega}}\otimes\mathbb{C}\,)\times(F_\omega\otimes\mathbb{C}\,)) \simeq \mathrm{GL}_{n_\ell[f_\ell\cdot\ell]}((F_{\overline{\omega}}\otimes\mathbb{C}\,)\times(F_\omega\otimes\mathbb{C}\,))$ *where* $G^{(2n_\ell[2f_\ell\cdot\ell])}((F_{\overline{\omega}}\otimes\mathbb{C}\,)\times(F_\omega\otimes\mathbb{C}\,))$ *is a condensed notation for the shifted bilinear complete semigroup* $M_R^{2n_\ell}[2f_\ell\cdot\ell]\otimes M_L^{2n_\ell}[2f_\ell\cdot\ell]$ *;*
2) $\mathrm{Irr}\,\mathrm{Rep}^{(2n_\ell[2f_\ell\cdot\ell])}_{W_{F_{R\times L}}}(W^{ab}_{F^{S_\mathbb{C}}_{\overline{\omega}}} \times W^{ab}_{F^{S_\mathbb{C}}_{\omega}}) = G^{(2n_\ell[2f_\ell\cdot\ell])}((F_{\overline{\omega}_\oplus}\otimes\mathbb{C}\,)\times(F_{\omega_\oplus}\otimes\mathbb{C}\,))$ *where* $\mathrm{Irr}\,\mathrm{Rep}^{(2n_\ell[2f_\ell\cdot\ell])}_{W_{F_{R\times L}}}(W^{ab}_{F^{S_\mathbb{C}}_{\overline{\omega}}} \times W^{ab}_{F^{S_\mathbb{C}}_{\omega}})$ *is the irreducible* $2n_\ell$ *-dimensional shifted global Weil-Deligne representation of* $(W^{ab}_{F^{S_\mathbb{C}}_{\overline{\omega}}} \times W^{ab}_{F^{S_\mathbb{C}}_{\omega}})$ *.*

Proof.
- The shifted bilinear complete semigroup $G^{(2n_\ell[2f_\ell\cdot\ell])}((F_{\overline{\omega}}\otimes\mathbb{C}\,)\times(F_\omega\otimes\mathbb{C}\,)) = \mathrm{Repsp}(\mathrm{GL}_{n_\ell[f_\ell\cdot\ell]}((F_{\overline{\omega}}\otimes\mathbb{C}\,)\times(F_\omega\otimes\mathbb{C}\,)))$, isomorphic to the bilinear semigroup of matrices $\mathrm{GL}_{n_\ell[f_\ell\cdot\ell]}((F_{\overline{\omega}}\otimes\mathbb{C}\,)\times(F_\omega\otimes\mathbb{C}\,))$, is a $\mathrm{GL}_{n_\ell[f_\ell\cdot\ell]}((F_{\overline{\omega}}\otimes\mathbb{C}\,)\times(F_\omega\otimes\mathbb{C}\,))$-bisemimodule $M_R^{2n_\ell}[2f_\ell\cdot\ell]\otimes M_L^{2n_\ell}[2f_\ell\cdot\ell]$.
- The representation of $\mathrm{GL}_{n_\ell[f_\ell\cdot\ell]}((F_{\overline{\omega}}\otimes\mathbb{C}\,)\times(F_\omega\otimes\mathbb{C}\,))$ into $M_R^{2n_\ell}[2f_\ell\cdot\ell]\otimes M_L^{2n_\ell}[2f_\ell\cdot\ell]$ corresponds to an algebraic morphism from $\mathrm{GL}_{n_\ell[f_\ell\cdot\ell]}((F_{\overline{\omega}}\otimes\mathbb{C}\,)\times(F_\omega\otimes\mathbb{C}\,))$ into $\mathrm{GL}(M_R^{2n_\ell}[2f_\ell\cdot\ell]\otimes M_L^{2n_\ell}[2f_\ell\cdot\ell])$ which denotes the group of automorphisms of $M_R^{2n_\ell}[2f_\ell\cdot\ell]\otimes M_L^{2n_\ell}[2f_\ell\cdot\ell]$.

  So, $\mathrm{GL}(M_R^{2n_\ell}[2f_\ell\cdot\ell]\otimes M_L^{2n_\ell}[2f_\ell\cdot\ell])$ constitutes the $2n_\ell$-dimensional equivalent of the product $(W^{ab}_{F^{S_\mathbb{C}}_{\overline{\omega}}} \times W^{ab}_{F^{S_\mathbb{C}}_{\omega}})$ of shifted global Weil groups.
- As $\mathrm{GL}(M_R^{2n_\ell}[2f_\ell\cdot\ell]\otimes M_L^{2n_\ell}[2f_\ell\cdot\ell])$ is isomorphic to $\mathrm{GL}_{n_\ell[f_\ell\cdot\ell]}((F_{\overline{\omega}}\otimes\mathbb{C}\,)\times(F_\omega\otimes\mathbb{C}\,))$, the shifted bilinear semigroup $G^{(2n_\ell[2f_\ell\cdot\ell])}((F_{\overline{\omega}_\oplus}\otimes\mathbb{C}\,)\times(F_{\omega_\oplus}\otimes\mathbb{C}\,))$ becomes the natural irreducible $2n_\ell$-dimensional shifted global Weil-Deligne representation of $(W^{ab}_{F^{S_\mathbb{C}}_{\overline{\omega}}} \times W^{ab}_{F^{S_\mathbb{C}}_{\omega}})$. ■

5.2 Proposition

On the shifted bilinear complete semigroup $G^{(2n_\ell[2f_\ell\cdot\ell])}((F_{\overline{\omega}}\otimes\mathbb{C}\,)\times(F_\omega\otimes\mathbb{C}\,))$ *there exists the* ***geometric-shifted global bilinear correspondence of Langlands:***

$$\begin{tikzcd}
\mathrm{IrrRep}^{(2n_\ell[2f_\ell\cdot\ell])}_{W_{F_{R\times L}}}(W^{ab}_{F^{S_\mathbb{C}}_{\overline{\omega}}}\times W^{ab}_{F^{S_\mathbb{C}}_{\omega}}\,) \arrow[r, "\sim"] \arrow[d, equal] & \mathrm{Irrcusp}(\mathrm{GL}_{n_\ell[f_\ell\cdot\ell]}((F^T_{\overline{\omega}}\otimes\mathbb{C}\,)\times(F^T_\omega\otimes\mathbb{C}\,))) \arrow[d, equal] \\
G^{(2n_\ell[2f_\ell\cdot\ell])}((F_{\overline{\omega}_\oplus}\otimes\mathbb{C}\,)\times(F_{\omega_\oplus}\otimes\mathbb{C}\,)) \arrow[dr, "\sim"] & \mathrm{EIS}_{R\times L}(2n_\ell[2f_\ell\cdot\ell],j,m_j) \\
& G^{(2n_\ell[2f_\ell\cdot\ell])}((F^T_{\overline{\omega}}\otimes\mathbb{C}\,)\times(F^T_\omega\otimes\mathbb{C}\,)) \arrow[u, "\wr"] \arrow[d, phantom, "\wr"] \\
& H^{2n_\ell-2f_\ell\cdot\ell}(\overline{S}^{P_n[f_\ell\cdot\ell]}_{\mathrm{GL}_n[f_\ell\cdot\ell]},\widehat{M}^{2n_\ell}_{T_{R_\oplus}}[2f_\ell\cdot\ell]\otimes\widehat{M}^{2n_\ell}_{T_{L_\oplus}}[2f_\ell\cdot\ell]) \arrow[d, phantom, "\wr"] \\
& \mathrm{CY}^{2n_\ell}_\oplus(Y_R,[2f_\ell\cdot\ell])\times\mathrm{CY}^{2n_\ell}_\oplus(Y_L,[2f_\ell\cdot\ell])
\end{tikzcd}$$

- *from the (infinite) sum of products, right by left, of the equivalence classes of the irreducible* $2n_\ell$*-dimensional shifted global Weil-Deligne representation* $\mathrm{IrrRep}^{(2n_\ell[2f_\ell\cdot\ell])}_{W_{F_{R\times L}}}(W^{ab}_{F^{S_\mathbb{C}}_{\overline{\omega}}}\times W^{ab}_{F^{S_\mathbb{C}}_{\omega}}\,)$ *of the shifted bilinear global Weil group* $W^{ab}_{F^{S_\mathbb{C}}_{\overline{\omega}}}\times W^{ab}_{F^{S_\mathbb{C}}_{\omega}}$ *given by the shifted bilinear complete semigroup* $G^{(2n_\ell[2f_\ell\cdot\ell])}((F_{\overline{\omega}_\oplus}\otimes\mathbb{C}\,)\times(F_{\omega_\oplus}\otimes\mathbb{C}\,))$
- *to the shifted irreducible supercuspidal representation* $\mathrm{Irrcusp}(\mathrm{GL}_{n_\ell[f_\ell\cdot\ell]}((F^T_{\overline{\omega}}\otimes\mathbb{C}\,)\times(F^T_\omega\otimes\mathbb{C}\,)))$ *of* $\mathrm{GL}_{n_\ell[f_\ell\cdot\ell]}((F^T_{\overline{\omega}}\otimes\mathbb{C}\,)\times(F^T_\omega\otimes\mathbb{C}\,))$ *given by the* $2n_\ell$*-dimensional solvable truncated shifted Eisenstein biserie* $\mathrm{EIS}_{R\times L}(2n_\ell[2f_\ell\cdot\ell],j,m_j)$
- *which are in one-to-one correspondence with the (infinite) sum of the products, right by left, of the equivalence classes of the irreducible* $2n_\ell$*-dimensional shifted cycle representatives* $\mathrm{CY}^{2n_\ell}_\oplus(Y_R,[2f_\ell\cdot\ell])\times\mathrm{CY}^{2n_\ell}_\oplus(Y_L,[2f_\ell\cdot\ell])$.

Proof.
- This proposition is an adaptation of proposition 3.4.14 of [Pie3] to the shifted case.
- In lemma 5.1, it was proved that $\mathrm{IrrRep}^{(2n_\ell[2f_\ell\cdot\ell])}_{W_{F_{R\times L}}}(W^{ab}_{F^{S_\mathbb{C}}_{\overline{\omega}}}\times W^{ab}_{F^{S_\mathbb{C}}_{\omega}}\,)=G^{(2n_\ell[2f_\ell\cdot\ell])}((F_{\overline{\omega}_\oplus}\otimes\mathbb{C}\,)\times(F_{\omega_\oplus}\otimes\mathbb{C}\,))$ in such a way that:
  a) $G^{(2n_\ell[2f_\ell\cdot\ell])}((F^T_{\overline{\omega}_\oplus}\otimes\mathbb{C}\,)\times(F^T_{\omega_\oplus}\otimes\mathbb{C}\,))\approx G^{(2n_\ell[2f_\ell\cdot\ell])}((F_{\overline{\omega}_\oplus}\otimes\mathbb{C}\,)\times(F_{\omega_\oplus}\otimes\mathbb{C}\,))$ as mentioned in the proof of proposition 4.9.

b) $G^{(2n_\ell[2f_\ell\cdot\ell])}((F^T_{\overline{\omega}_\oplus}\otimes\mathbb{C})\times(F^T_{\omega_\oplus}\otimes\mathbb{C}))=(M^{2n_\ell}_{T_{R_\oplus}}[2f_\ell\cdot\ell]\otimes M^{2n_\ell}_{T_{L_\oplus}}[2f_\ell\cdot\ell])$ has an analytical development given by $\mathrm{EIS}_{R\times L}(2n_\ell[2f_\ell\cdot\ell],j,m_j)$ according to theorem 4.9 which leads to the bijection:

$$G^{(2n_\ell[2f_\ell\cdot\ell])}((F_{\overline{\omega}_\oplus}\otimes\mathbb{C})\times(F_{\omega_\oplus}\otimes\mathbb{C}))\xrightarrow{\ \sim\ }\mathrm{EIS}_{R\times L}(2n_\ell[2f_\ell\cdot\ell],j,m_j)\ .$$

- Finally, from proposition 4.5, it results that:

$$G^{(2n_\ell[2f_\ell\cdot\ell])}((F^T_{\overline{\omega}}\otimes\mathbb{C})\times(F^T_{\omega}\otimes\mathbb{C})))\approx\mathrm{CY}^{2n_\ell}_{\oplus}(Y_R,[2f_\ell\cdot\ell])\times\mathrm{CY}^{2n_\ell}_{\oplus}(Y_L,[2f_\ell\cdot\ell])$$

where

$$\mathrm{CY}^{2n_\ell}_{\oplus}(Y_L,[2f_\ell\cdot\ell])=\bigoplus_j\bigoplus_{m_j}\mathrm{CY}^{2n_\ell}_{\oplus}(Y_L,[2f_\ell\cdot\ell])\ . \qquad \blacksquare$$

5.3 Definition: Partially reducible shifted representation

Similarly as in [Pie3], shifted global reducible correspondences of Langlands can be introduced.

Let us, for example, consider a partition $n=n_1+\cdots+n_\ell+\cdots+n_s$ of n leading to the shifted partition:

$$n[f_n\cdot\ell]=n_1[f_1\cdot\ell]+\cdots+n_\ell[f_\ell\cdot\ell]+\cdots+n_s[f_s\cdot\ell]\ ,$$

where it can be assumed that $f_n=f_1+\cdots+f_\ell+\cdots+f_s$, in such a way that:

$$\begin{aligned}&\mathrm{Rep}(\mathrm{GL}_{n[f_n\cdot\ell]=n_1[f_1\cdot\ell]+\cdots+n_s[f_s\cdot\ell]}((F_{\overline{\omega}}\otimes\mathbb{C})\times(F_\omega\otimes\mathbb{C})))\\&\qquad=\boxplus_{n_\ell=n_1}^{n_s}\mathrm{Irr}\,\mathrm{Rep}(\mathrm{GL}_{n_\ell[f_\ell\cdot\ell]}((F_{\overline{\omega}}\otimes\mathbb{C})\times(F_\omega\otimes\mathbb{C})))\end{aligned}$$

constitutes a partially reducible shifted representation of $\mathrm{GL}_{n[f_n\cdot\ell]}((F_{\overline{\omega}}\otimes\mathbb{C})\times(F_\omega\otimes\mathbb{C}))$.

5.4 Proposition

If

$$G^{(2n[2f_n\cdot\ell])}((F_{\overline{\omega}}\otimes\mathbb{C})\times(F_\omega\otimes\mathbb{C}))=\boxplus_{n_\ell=n_1}^{n_s}G^{(2n_\ell[2f_\ell\cdot\ell])}((F_{\overline{\omega}}\otimes\mathbb{C})\times(F_\omega\otimes\mathbb{C}))$$

represents the decomposition of the shifted $2n$-dimensional bilinear complete semigroup into irreducible components of dimension $2n_\ell$ shifted in $2f_\ell\cdot\ell$ dimensions, then we have that:

$$\begin{aligned}G^{(2n[2f_n\cdot\ell])}((F_{\overline{\omega}}\otimes\mathbb{C})\times(F_\omega\otimes\mathbb{C}))&=\mathrm{RedRep}^{(2n[2f_n\cdot\ell])}_{W_{F_{R\times L}}}(W^{ab}_{F^{S_\mathbb{C}}_{\overline{\omega}}}\times W^{ab}_{F^{S_\mathbb{C}}_{\omega}})\\&=\bigoplus_{n_\ell=n_1}^{n_s}\mathrm{Irr}\,\mathrm{Rep}^{(2n_\ell[2f_\ell\cdot\ell])}_{W_{F_{R\times L}}}(W^{ab}_{F^{S_\mathbb{C}}_{\overline{\omega}}}\times W^{ab}_{F^{S_\mathbb{C}}_{\omega}})\end{aligned}$$

where $\text{RedRep}^{(2n[2f_\ell\cdot\ell])}_{W_{F_{R\times L}}}(W^{ab}_{F^{S_{\mathbb{C}}}_{\overline{\omega}}}\times W^{ab}_{F^{S_{\mathbb{C}}}_{\omega}})$ *denotes the* $2n$ *-dimensional reducible shifted global Weil-Deligne representation of the shifted bilinear global Weil group* $(W^{ab}_{F^{S_{\mathbb{C}}}_{\overline{\omega}}}\times W^{ab}_{F^{S_{\mathbb{C}}}_{\omega}})$ *.*

Proof. According to lemma 5.1, we have that:

$$\text{Irr}\,\text{Rep}^{(2n_\ell[2f_\ell\cdot\ell])}_{W_{F_{R\times L}}}(W^{ab}_{F^{S_{\mathbb{C}}}_{\overline{\omega}}}\times W^{ab}_{F^{S_{\mathbb{C}}}_{\omega}})=G^{(2n_\ell[2f_\ell\cdot\ell])}((F_{\overline{\omega}}\otimes\mathbb{C})\times(F_\omega\otimes\mathbb{C}))$$

from which the thesis follows if definition 5.3 is taken into account. ■

5.5 Proposition

The toroidal compactification of $G^{(2n[2f_n\cdot\ell])}((F_{\overline{\omega}_\oplus}\otimes\mathbb{C})\times(F_{\omega_\oplus}\otimes\mathbb{C}))$ *generates*

$$G^{(2n[2f_n\cdot\ell])}((F^T_{\overline{\omega}_\oplus}\otimes\mathbb{C})\times(F^T_{\omega_\oplus}\otimes\mathbb{C}))=\bigoplus_{n_\ell=n_1}^{n_s}G^{2n_\ell[2f_\ell\cdot\ell]}((F^T_{\overline{\omega}_\oplus}\otimes\mathbb{C})\times(F^T_{\omega_\oplus}\otimes\mathbb{C}))$$

whose supercuspidal representation is given by:

$$\text{Redcusp}(\text{GL}_{n[f_n\cdot\ell]}((F^T_{\overline{\omega}}\otimes\mathbb{C})\times(F^T_{\omega}\otimes\mathbb{C})))=\bigoplus_{n_\ell=n_1}^{n_s}\text{EIS}_{R\times L}(2n_\ell[2f_\ell\cdot\ell],j,m_j)$$

where $\text{EIS}_{R\times L}(2n_\ell[2f_\ell\cdot\ell],j,m_j)$ *is the* $2n_\ell$ *-dimensional truncated shifted cuspidal biserie.*

Proof. This directly results from definition 5.3 and proposition 4.13. ■

5.6 Proposition

Let $\overline{S}^{P_{n[f_n\cdot\ell]}}_{\text{GL}_{n[f_n\cdot\ell]}}=\boxplus_{n_\ell=n_1}^{n_s}\overline{S}^{P_{n_\ell[f_\ell\cdot\ell]}}_{\text{GL}_{n_\ell[f_\ell\cdot\ell]}}$ *be the decomposition of the reducible shifted* $2n$ *-dimensional bisemispace* $\overline{S}^{P_{n[f_n\cdot\ell]}}_{\text{GL}_{n[f_n\cdot\ell]}}$ *into irreducible components* $\overline{S}^{P_{n_\ell[f_\ell\cdot\ell]}}_{\text{GL}_{n_\ell[f_\ell\cdot\ell]}}$ *introduced in section 4.1.*
Then, the cohomology of this reducible shifted bisemispace $\overline{S}^{P_{n[f_n\cdot\ell]}}_{\text{GL}_{n[f_n\cdot\ell]}}$ *decomposes according to:*

$$\begin{aligned}
&H^*(\overline{S}^{P_{n[f_n\cdot\ell]}}_{\text{GL}_{n[f_n\cdot\ell]}},\widehat{M}^{2n}_{T_R}[2f_n\cdot\ell]\otimes\widehat{M}^{2n}_{T_L}[2f_n\cdot\ell])\\
&\qquad=\bigoplus_{n_\ell=n_1}^{n_s}H^{2n_\ell-2f_\ell\cdot\ell}(\overline{S}^{P_{n[f_n\cdot\ell]}}_{\text{GL}_{n[f_n\cdot\ell]}},\widehat{M}^{2n_\ell}_{T_R}[2f_\ell\cdot\ell]\otimes\widehat{M}^{2n_\ell}_{T_L}[2f_\ell\cdot\ell])\\
&\qquad\approx\bigoplus_{n_\ell}\text{CY}^{2n_\ell}(Y_R,[2f_\ell\cdot\ell])\times\text{CY}^{2n}(Y_L,[2f_\ell\cdot\ell])
\end{aligned}$$

where $(\widehat{M}^{2n}_{T_R}[2f_n\cdot\ell]\otimes\widehat{M}^{2n_\ell}_{T_L}[2f_n\cdot\ell])$ *is the bisemisheaf over the "partially reducible"* $\text{GL}_{n[f_n\cdot\ell]}((F^T_{\overline{\omega}}\otimes\mathbb{C})\times(F^T_{\omega}\otimes\mathbb{C}))$ *-bisemimodule.*

5.7 Proposition

On the reducible shifted pseudoramified bilinear complete semigroup

$$G^{2n[2f_n\cdot\ell]}((F_{\overline{\omega}}\otimes\mathbb{C}\,)\times(F_\omega\otimes\mathbb{C}\,))=\mathop{\boxplus}_{n_\ell=n_1}^{n_s}G^{2n_\ell[2f_\ell\cdot\ell]}((F_{\overline{\omega}}\otimes\mathbb{C}\,)\times(F_\omega\otimes\mathbb{C}\,))\;,$$

there exists the ***geometric-shifted global bilinear "reducible" correspondence of Langlands:***

$$\begin{tikzcd}
\mathrm{RedRep}^{(2n[2f_n\cdot\ell])}_{W_{F_{R\times L}}}(W^{ab}_{F^{S_{\mathbb{C}}}_{\overline{\omega}}}\times W^{ab}_{F^{S_{\mathbb{C}}}_{\omega}}) \arrow[r, "\sim"] \arrow[d, equal] & \mathrm{Redcusp}(\mathrm{GL}_{n[f_n\cdot\ell]}((F^T_{\overline{\omega}}\otimes\mathbb{C}\,)\times(F^T_\omega\otimes\mathbb{C}\,))) \arrow[d, equal] \\
G^{(2n[2f_n\cdot\ell])}((F_{\overline{\omega}_\oplus}\otimes\mathbb{C}\,)\times(F_{\omega_\oplus}\otimes\mathbb{C}\,)) \arrow[r, "\sim"] \arrow[dr, "\sim"'] & \bigoplus\limits_{n_\ell}\mathrm{EIS}_{R\times L}(2n_\ell[2f_\ell\cdot\ell],j,m_j) \\
 & G^{(2n[2f_n\cdot\ell])}((F^T_{\overline{\omega}}\otimes\mathbb{C}\,)\times(F^T_\omega\otimes\mathbb{C}\,)) \arrow[u, "\wr"] \arrow[d, "\wr"] \\
 & H^*(\overline{S}^{P_{n[f_n\cdot\ell]}}_{\mathrm{GL}_{n[f_n\cdot\ell]}},\widehat{M}^{2n}_{T_{R_\oplus}}[2f_n\cdot\ell]\otimes\widehat{M}^{2n}_{T_{L_\oplus}}[2f_n\cdot\ell]) \arrow[d, phantom, "\wr\wr"] \\
 & \bigoplus\limits_{n_\ell}(\mathrm{CY}^{2n_\ell}_{\oplus}(Y_R,[2f_\ell\cdot\ell])\times\mathrm{CY}^{2n_\ell}_{\oplus}(Y_L,[2f_\ell\cdot\ell]))
\end{tikzcd}$$

■

The objective consists now in establishing shifted global bilinear correspondences on the boundary $\partial\overline{S}^{P_{n[f_n\cdot\ell]}}_{\mathrm{GL}_{n[f_n\cdot\ell]}}$ of the shifted bisemispace $\overline{S}^{P_{n[f_n\cdot\ell]}}_{\mathrm{GL}_{n[f_n\cdot\ell]}}$ and to introduce by this way shifted Eisenstein cohomology.

5.8 Shifted real completions and shifted global Weil groups

- From sections 1.2 and 1.4, **direct sums of shifted real completions** will be given by:

$$F^{+,(S_{\mathbb{R}})}_{v_\oplus}=\bigoplus_{j_\delta}\bigoplus_{m_{j_\delta}}(F^+_{v_{j_\delta,m_{j_\delta}}}\otimes\mathbb{R}\,)\qquad(\text{resp.}\quad F^{+,(S_{\mathbb{R}})}_{\overline{v}_\oplus}=\bigoplus_{j_\delta}\bigoplus_{m_{j_\delta}}(F^+_{\overline{v}_{j_\delta,m_{j_\delta}}}\otimes\mathbb{R}\,)\;).$$

- According to [Pie3], there is a one-to-one correspondence between the real shifted completions and their complex equivalents.

- **The global Weil groups** $W^{ab}_{F_v^{+,(S_{\mathbb{R}})}}$ (resp. $W^{ab}_{F_{\overline{v}}^{+,(S_{\mathbb{R}})}}$), shifted over $\mathbb{R}$ and referring to pseudoramified extensions characterized by degrees $d=0 \bmod N$, will be introduced by:

$$W^{ab}_{F_v^{+,(S_{\mathbb{R}})}}=\underset{j_\delta,m_{j_\delta}}{\oplus}\mathrm{Gal}(\dot{\widetilde{F}}^{+,(S_{\mathbb{R}})}_{v_{j_\delta,m_{j_\delta}}}/k)\qquad(\text{resp.}\quad W^{ab}_{F_{\overline{v}}^{+,(S_{\mathbb{R}})}}=\underset{j_\delta,m_{j_\delta}}{\oplus}\mathrm{Gal}(\dot{\widetilde{F}}^{+,(S_{\mathbb{R}})}_{\overline{v}_{j_\delta,m_{j_\delta}}}/k)\ )$$

where $\dot{\widetilde{F}}^{+,(S_{\mathbb{R}})}_{\overline{v}_{j_\delta,m_{j_\delta}}}$ denote the shifted pseudoramified extensions with degrees $d=0\bmod(N)$.

5.9 The boundary of the Borel-Serre compactification shifted over $\mathbb{R}$

- As developed in [Pie3], the boundary $\partial\overline{Y}^{(2n_\ell)}_{S^T_{R\times L}}$ of the Borel-Serre compactification $\overline{Y}^{(2n_\ell)}_{S^T_{R\times L}}=\mathrm{GL}_{n_\ell}(F^T_R\times F^T_L)/\mathrm{GL}_{n_\ell}((\mathbb{Z}/N\,\mathbb{Z})^2)$ of $Y^{(2n_\ell)}_{S_{R\times L}}$ (see section 1.12) is given by:

$$\begin{aligned}\partial\overline{Y}^{(2n_\ell)}_{S^T_{R\times L}}&=\mathrm{GL}_{n_\ell}(F^{+,T}_R\times F^{+,T}_L)/\mathrm{GL}_{n_\ell}((\mathbb{Z}/N\,\mathbb{Z})^2)\\&=\mathrm{GL}_{n_\ell}(F^{+,T}_{\overline{v}}\times F^{+,T}_v)\end{aligned}$$

where:

- $F^{+,T}_R$ and $F^{+,T}_L$ are real toroidal compactifications of $\widetilde{F}^+_R$ and $\widetilde{F}^+_L$ respectively;
- $F^{+,T}_v=\{F^{+,T}_{v_1},\dots,F^{+,T}_{v_{j_\delta,m_{j_\delta}}},\dots,F^{+,T}_{v_{r_\delta,m_{r_\delta}}}\}$ is the set of real toroidal completions.

The boundary $\boldsymbol{\partial\overline{Y}^{(2n_\ell)}_{S^T_{R\times L}}}$ shifted over $\mathbb{R}$ in $\boldsymbol{(2f_\ell\cdot\ell)}$ real dimensions is given by:

$$\begin{aligned}\partial\overline{Y}^{2n_\ell[2f_\ell\cdot\ell]}_{S^T_{R\times L}}&=\mathrm{GL}_{n_\ell[f_\ell\cdot\ell]}((F^{+,T}_R\otimes\mathbb{R})\times(F^{+,T}_L\otimes\mathbb{R}))/\mathrm{GL}_{n_\ell[f_\ell\cdot\ell]}((\mathbb{Z}/N\,\mathbb{Z})^2\otimes\mathbb{R}^2)\\&\approx\mathrm{GL}_{n_\ell[f_\ell\cdot\ell]}((F^{+,T}_{\overline{v}}\otimes\mathbb{R})\times(F^{+,T}_v\otimes\mathbb{R}))\,.\end{aligned}$$

- The corresponding **double coset decomposition of the shifted bilinear semigroup** $\mathrm{GL}_{n_\ell[f_\ell\cdot\ell]}((F^{+,T}_{\overline{v}}\otimes\mathbb{R})\times(F^{+,T}_v\otimes\mathbb{R}))$ can be introduced by:

$$\begin{aligned}\partial\overline{S}^{P_{n_\ell[f_\ell\cdot\ell]}}_{\mathrm{GL}_{n_\ell[f_\ell\cdot\ell]}}&=P_{n_\ell[f_\ell\cdot\ell]}((F^{+,T}_{\overline{v}^1}\otimes\mathbb{R})\times(F^{+,T}_{\overline{v}^1}\otimes\mathbb{R}))\\&\qquad\backslash\,\mathrm{GL}_{n_\ell[f_\ell\cdot\ell]}((F^{+,T}_R\otimes\mathbb{R})\times(F^{+,T}_L\otimes\mathbb{R}))/\mathrm{GL}_{n_\ell[f_\ell\cdot\ell]}((\mathbb{Z}/N\,\mathbb{Z})^2\otimes\mathbb{R}^2)\end{aligned}$$

where $F^{+,T}_{\overline{v}^1}$ and $F^{+,T}_{v^1}$ denote the set of irreducible subcompletions characterized by a degree N (see section 1.1).

5.10 Action of the differential bioperator $(D_R^{2f_\ell\cdot\ell}\otimes D_L^{2f_\ell\cdot\ell})$

The differential bioperator $(D_R^{2f_\ell\cdot\ell}\otimes D_L^{2f_\ell\cdot\ell})$ maps the bisemisheaf $(\widehat M^{2n_\ell}_{T_{v_R}}\otimes \widehat M^{2n_\ell}_{T_{v_L}})$ over the $\mathrm{GL}_{n_\ell}(F^{+,T}_{\overline v}\times (F^{+,T}_{\overline v})$-bisemimodule $(M^{2n_\ell}_{T_{v_R}}\otimes M^{2n_\ell}_{T_{v_L}})$ into the corresponding perverse bisemisheaf $(\widehat M^{2n_\ell}_{T_{v_R}}[2f_\ell\cdot\ell]\otimes \widehat M^{2n_\ell}_{T_{v_L}}[2f_\ell\cdot\ell])$ over the $\mathrm{GL}_{n_\ell[f_\ell\cdot\ell]}(F^{+,T}_{\overline v}\times\mathbb{R}\,)\times(F^{+,T}_{v}\times\mathbb{R}\,))$-bisemimodule $(M^{2n_\ell}_{T_{v_R}}[2f_\ell\cdot\ell]\otimes M^{2n_\ell}_{T_{v_L}}[2f_\ell\cdot\ell])$ according to

$$D_R^{2f_\ell\cdot\ell}\otimes D_L^{2f_\ell\cdot\ell}:\qquad \widehat M^{2n_\ell}_{T_{v_R}}\otimes \widehat M^{2n_\ell}_{T_{v_L}}\longrightarrow \widehat M^{2n_\ell}_{T_{v_R}}[2f_\ell\cdot\ell]\otimes \widehat M^{2n_\ell}_{T_{v_L}}[2f_\ell\cdot\ell]\;.$$

5.11 Proposition

The bilinear cohomology of the shifted Shimura bisemivariety $\partial\overline S^{P_{n[f_n\cdot\ell]}}_{\mathrm{GL}_{n[f_n\cdot\ell]}}$, $n>n_\ell$, is the bilinear shifted Eisenstein cohomology:

$$\begin{aligned}H^{2n_\ell-2f_\ell\cdot\ell}(\partial\overline S^{P_{n[f_n\cdot\ell]}}_{\mathrm{GL}_{n[f_n\cdot\ell]}}),\widehat M^{2n_\ell}_{T_{v_R}}[2f_\ell\cdot\ell]\otimes \widehat M^{2n_\ell}_{T_{v_L}}[2f_\ell\cdot\ell])&\\ \simeq \mathrm{FRepsp}(\mathrm{GL}_{n_\ell[f_\ell\cdot\ell]}((F^{+,T}_{\overline v}\otimes\mathbb{R}\,)\times(F^{+,T}_{v}\otimes\mathbb{R}\,))&\end{aligned}$$

in such a way that the functional representation space $\mathrm{FRepsp}(\mathrm{GL}_{n_\ell[f_\ell\cdot\ell]}((F^{+,T}_{\overline v}\otimes\mathbb{R}\,)\times(F^{+,T}_{v}\otimes\mathbb{R}\,))$ *of the shifted complete bilinear semigroup* $\mathrm{GL}_{n_\ell[f_\ell\cdot\ell]}((F^{+,T}_{\overline v}\otimes\mathbb{R}\,)\times(F^{+,T}_{v}\otimes\mathbb{R}\,)$ *is:*

$$\begin{aligned}&\mathrm{FRepsp}(\mathrm{GL}_{n_\ell[f_\ell\cdot\ell]}((F^{+,T}_{\overline v}\otimes\mathbb{R}\,)\times(F^{+,T}_{v}\otimes\mathbb{R}\,))\\ &\qquad\equiv \widehat G^{2n_\ell[2f_\ell\cdot\ell]}((F^{+,T}_{\overline v}\otimes\mathbb{R}\,)\times(F^{+,T}_{v}\otimes\mathbb{R}\,))\\ &\qquad=\{(\phi(g^{n_\ell}_{T_R}([f_\ell\cdot\ell],(j_\delta,m_{j_\delta}))\times\phi(g^{n_\ell}_{T_L}([f_\ell\cdot\ell],(j_\delta,m_{j_\delta})))\}_{j_\delta,m_{j_\delta}}\\ &\qquad\simeq\{(\mathrm{CY}^{n_\ell}(\partial\overline Y_R,[f_\ell\cdot\ell]),(j_\delta,m_{j_\delta})\times\mathrm{CY}^{n_\ell}(\partial\overline Y_L,[f_\ell\cdot\ell]),(j_\delta,m_{j_\delta})\}_{j_\delta,m_{j_\delta}}\end{aligned}$$

where:

- $\widehat G^{2n_\ell[2f_\ell\cdot\ell]}((F^{+,T}_{\overline v}\otimes\mathbb{R}\,)\times(F^{+,T}_{v}\otimes\mathbb{R}\,))$ *is the bisemisheaf over* $G^{2n_\ell[2f_\ell\cdot\ell]}((F^{+,T}_{\overline v}\otimes\mathbb{R}\,)\times(F^{+,T}_{v}\otimes\mathbb{R}\,))$ *which decomposes according to the set of products, right by left,* $g^{n_\ell}_{T_{R\times L}}([f_\ell\cdot\ell],(j_\delta,m_{j_\delta}))$ *of conjugacy class representatives of real dimension* n_ℓ *.*
- *the functions* $\phi(g^{n_\ell}_{T_L}([f_\ell\cdot\ell],(j_\delta,m_{j_\delta}))$ *on these conjugacy class representatives are in one-to-one correspondence with the equivalent class representative* $\mathrm{CY}^{n_\ell}(\partial\overline Y_L,[f_\ell\cdot\ell]),(j_\delta,m_{j_\delta}))$ *of cycles of codimension* n_ℓ *shifted in* $f_\ell\cdot\ell$ *real dimension.*

Proof. This is an adaptation to the real case of propositions 4.5 and 4.6. ∎

5.12 Shifted global elliptic semimodules

- Every left (resp. right) function on the conjugacy class representative $g^{n_\ell}_{T_L}([f_\ell \cdot \ell],(j_\delta,m_{j_\delta}))$ (resp. $g^{n_\ell}_{T_R}([f_\ell \cdot \ell],(j_\delta,m_{j_\delta}))$) of $\mathrm{GL}_{n_\ell[f_\ell\cdot\ell]}((F^{+,T}_{\overline{v}}\otimes\mathbb{R})\times(F^{+,T}_{v}\otimes\mathbb{R}))$ is a n_ℓ-dimensional real semitorus $T^{n_\ell}_L([f_\ell \cdot \ell],(j_\delta,m_{j_\delta}))$ (resp. $T^{n_\ell}_R([f_\ell \cdot \ell],(j_\delta,m_{j_\delta}))$) shifted in $f_\ell\cdot\ell$ real dimensions, localized in the upper (resp. lower) half space and having the following analytic development:

$$T^{n_\ell}_L([f_\ell \cdot \ell],(j_\delta,m_{j_\delta})) \simeq E_{f_\ell\cdot\ell}(n_\ell,j_\delta,m_{j_\delta})\ \lambda^{\frac{1}{2}}(n_\ell,j_\delta,m_{j_\delta})\ e^{2\pi i j_\delta x_{n_\ell}}$$

$$(\text{resp.}\quad T^{n_\ell}_R([f_\ell \cdot \ell],(j_\delta,m_{j_\delta})) \simeq E_{f_\ell\cdot\ell}(n_\ell,j_\delta,m_{j_\delta})\ \lambda^{\frac{1}{2}}(n_\ell,j_\delta,m_{j_\delta})\ e^{-2\pi i j_\delta x_{n_\ell}}\ )$$

  where:
  - $\lambda(n_\ell,j_\delta,m_{j_\delta})$ is a generalized Hecke global character obtained from the product of the eigenvalues of the (j_δ,m_{j_δ})-th coset representative of the Hecke bioperator (see proposition 4.7);
  - $E_{f_\ell\cdot\ell}(n_\ell,j_\delta,m_{j_\delta})$ is the shift in $f_\ell\cdot\ell$ real dimensions of $\lambda^{\frac{1}{2}}(n_\ell,j_\delta,m_{j_\delta})$;
  - $x_{n_\ell}=\sum\limits_{\beta=1}^{n_\ell} x_\beta\ \overrightarrow{e}^{\beta}$ is a vector of $\mathbb{R}^{n_\ell}$.

- The analytic representation of the shifted bilinear complete semigroup $G^{2n_\ell[2f_\ell\cdot\ell]}((F^{+,T}_{\overline{v}_\oplus}\otimes\mathbb{R})\times(F^{+,T}_{v_\otimes}\otimes\mathbb{R}))$, which is also a supercuspidal representation of $\mathrm{GL}_{n_\ell[f_\ell\cdot\ell]}((F^{+,T}_{\overline{v}}\otimes\mathbb{R})\times(F^{+,T}_{v}\otimes\mathbb{R}))$, is obtained by summing over j_δ and m_{j_δ} the analytic representations of the conjugacy class representatives $g^{n_\ell}_{T_{R\times L}}([f_\ell \cdot \ell],(j_\delta,m_{j_\delta}))$ giving rise to the product, right by left, of the shifted global elliptic semimodules [Drin]:

$$\begin{aligned}
&\mathrm{ELLIP}_R(2n_\ell[2f_\ell\cdot\ell],j_\delta,m_{j_\delta})\otimes\mathrm{ELLIP}_L(2n_\ell[2f_\ell\cdot\ell],j_\delta,m_{j_\delta})\\
&\quad=\left[\bigoplus_{j_\delta=1}^{r}\bigoplus_{m_{j_\delta}} E_{f_\ell\cdot\ell}(n_\ell,j_\delta,m_{j_\delta})\ \lambda^{\frac{1}{2}}(n_\ell,j_\delta,m_{j_\delta})\ e^{-2\pi i j_\delta x_{n_\ell}}\right]\\
&\qquad\otimes\left[\bigoplus_{j_\delta=1}^{r}\bigoplus_{m_{j_\delta}} E_{f_\ell\cdot\ell}(n_\ell,j_\delta,m_{j_\delta})\ \lambda^{\frac{1}{2}}(n_\ell,j_\delta,m_{j_\delta})\ e^{2\pi i j_\delta x_{n_\ell}}\right].
\end{aligned}$$

5.13 Proposition

The shifted bilinear Eisenstein cohomology has the following analytic development:

$$\begin{aligned}
&H^{2n_\ell-2f_\ell\cdot\ell}(\partial\overline{S}^{P_{n[f_n\cdot\ell]}}_{\mathrm{GL}_{n[f_n\cdot\ell]}}),\widehat{M}^{2n_\ell}_{T_{v_{R_\oplus}}}[2f_\ell\cdot\ell]\otimes\widehat{M}^{2n_\ell}_{T_{v_{L_\oplus}}}[2f_\ell\cdot\ell])\\
&\quad\simeq\mathrm{ELLIP}_R(2n_\ell[2f_\ell\cdot\ell],j_\delta,m_{j_\delta})\otimes\mathrm{ELLIP}_L(2n_\ell[2f_\ell\cdot\ell],j_\delta,m_{j_\delta})
\end{aligned}$$

and gives rise to the eigen(bi)value equation:

$$\begin{aligned}
&(D_R^{2f_\ell\cdot\ell}\otimes D_L^{2f_\ell\cdot\ell})(\mathrm{ELLIP}_R(2n_\ell,j_\delta^{\mathrm{up}}=j_\delta,m_{j_\delta})\otimes\mathrm{ELLIP}_L(2n_\ell,j_\delta^{\mathrm{up}}=j_\delta,m_{j_\delta}))\\
&\quad=E^2_{f_\ell\cdot\ell}(n_\ell,j_\delta,m_{j_\delta})_{\mathrm{eig}}(\mathrm{ELLIP}_R(2n_\ell,j_\delta^{\mathrm{up}}=j_\delta,m_{j_\delta})\otimes\mathrm{ELLIP}_L(2n_\ell,j_\delta^{\mathrm{up}}=j_\delta,m_{j_\delta}))\;.
\end{aligned}$$

Proof. This is an adaptation to the real case of proposition 4.9. ■

5.14 Proposition

Taking into account that the irreducible $2n_\ell$ *-dimensional shifted global Weil-Deligne representation* $\mathrm{Irr\,Rep}^{(2n_\ell[2f_\ell\cdot\ell])}_{W_{F^+_{R\times L}}}(W^{ab}_{F^{+,(S_{\mathbb{R}})}_{\overline{v}}}\times W^{ab}_{F^{+,(S_{\mathbb{R}})}_{v}})$ *of the shifted bilinear global Weil group* $(W^{ab}_{F^{+,(S_{\mathbb{R}})}_{\overline{v}}}\times W^{ab}_{F^{+,(S_{\mathbb{R}})}_{v}})$ *is given by the shifted bilinear complete semigroup* $G^{2n_\ell[2f_\ell\cdot\ell]}((F^{+,T}_{\overline{v}_\oplus}\otimes\mathbb{R})\times(F^{+,T}_{v_\oplus}\otimes\mathbb{R}))$ *, we have on the shifted Shimura bisemivariety* $\partial\overline{S}^{P_{n_\ell[f_\ell\cdot\ell]}}_{\mathrm{GL}_{n_\ell[f_\ell\cdot\ell]}}$ *the following* ***geometric-shifted global bilinear correspondence of Langlands*** [Lan]*:*

$$\begin{array}{ccc}
\mathrm{Irr\,Rep}^{(2n_\ell[2f_\ell\cdot\ell])}_{W_{F^+_{R\times L}}}(W^{ab}_{F^{+,(S_{\mathbb{R}})}_{\overline{v}}}\times W^{ab}_{F^{+,(S_{\mathbb{R}})}_{v}}) & \xrightarrow{\sim} & \mathrm{Irr\,ELLIP}(\mathrm{GL}_{n_\ell[f_\ell\cdot\ell]}((F^{+,T}_{\overline{v}}\otimes\mathbb{R})\times(F^{+,T}_{v}\otimes\mathbb{R})))\\
\| & & \|\\
G^{(2n_\ell[2f_\ell\cdot\ell])}((F^{+}_{\overline{v}_\oplus}\otimes\mathbb{R})\times(F^{+}_{v_\oplus})\otimes\mathbb{R}) & & \mathrm{ELLIP}_{R\times L}(2n_\ell[2f_\ell\cdot\ell],j_\delta,m_{j_\delta})\\
 & \searrow\sim & \uparrow\wr\\
 & & G^{(2n_\ell[2f_\ell\cdot\ell])}((F^{+,T}_{\overline{v}}\otimes\mathbb{R})\times(F^{+,T}_{v}\otimes\mathbb{R}))\\
 & & \wr\|\\
 & & H^{2n_\ell-2f_\ell\cdot\ell}(\partial\overline{S}^{P_{n[f_n\cdot\ell]}}_{\mathrm{GL}_{n[f_n\cdot\ell]}},\widehat{M}^{2n_\ell}_{T_{v_R}}[2f_\ell\cdot\ell]\otimes\widehat{M}^{2n_\ell}_{T_{v_L}}[2f_\ell\cdot\ell])\\
 & & \wr\wr\\
 & & \mathrm{CY}^{2n_\ell}(\partial\overline{Y}_R,[2f_\ell\cdot\ell])\times\mathrm{CY}^{2n_\ell}(\overline{Y}_L,[2f_\ell\cdot\ell])
\end{array}$$

where $\mathrm{Irr\,ELLIP}(\mathrm{GL}_{n_\ell[f_\ell\cdot\ell]}((F^{+,T}_{\overline{v}}\otimes\mathbb{R})\times(F^{+,T}_{v}\otimes\mathbb{R})))$ *is the shifted irreducible elliptic representation of* $\mathrm{GL}_{n_\ell[f_\ell\cdot\ell]}((F^{+,T}_{\overline{v}}\otimes\mathbb{R})\times(F^{+,T}_{v}\otimes\mathbb{R}))$ *given by the* $2n_\ell$ *-dimensional solvable elliptic bisemimodule* $\mathrm{ELLIP}_{R\times L}(2n_\ell[2f_\ell\cdot\ell],j_\delta,m_{j_\delta})$ *shifted in* $(2f_\ell\cdot\ell)$ *dimensions.*

5.15 The complex and real $2n_\ell$-dimensional irreducible geometric-shifted global bilinear correspondences of Langlands

can be summarized in the following diagram:

$$\begin{array}{ccc}
\mathrm{Irr}\,\mathrm{Rep}^{(2n_\ell[2f_\ell\cdot\ell])}_{W_{F_{R\times L}}}(W^{ab}_{F^{S_{\mathbb{C}}}_{\overline{\omega}}}\times W^{ab}_{F^{S_{\mathbb{C}}}_{\omega}}) & \longrightarrow & \mathrm{Irr}\,\mathrm{cusp}(\mathrm{GL}_{n_\ell[f_\ell\cdot\ell]}((F^T_{\overline{\omega}}\otimes\mathbb{C})\times(F^T_{\omega}\otimes\mathbb{C})))\\
\downarrow & & \downarrow\\
\mathrm{Irr}\,\mathrm{Rep}^{(2n_\ell[2f_\ell\cdot\ell])}_{W_{F^+_{R\times L}}}(W^{ab}_{F^{+,(S_{\mathbb{R}})}_{\overline{v}}}\times W^{ab}_{F^{(+,(S_{\mathbb{R}})}_{v}}) & \longrightarrow & \mathrm{Irr}\,\mathrm{ELLIP}(\mathrm{GL}_{n_\ell[f_\ell\cdot\ell]}((F^{+,T}_{\overline{v}}\otimes\mathbb{R})\times(F^{+,T}_{v}\otimes\mathbb{R})))
\end{array}$$

5.16 Definition

The partially reducible shifted representation of $\mathrm{GL}_{n[f_n\cdot\ell]}(F^+_{\overline{v}}\otimes\mathbb{R})\times(F^+_v\otimes\mathbb{R}))$ can be introduced, as in definition 5.3, on the basis of the shifted partition (real dimensions): $n\cdot[f_n\cdot\ell]=n_1[f_1\cdot\ell]+\cdots+n_\ell[f_\ell\cdot\ell]+\cdots+n_s[f_s\cdot\ell]$ by:

$$\begin{aligned}
&\mathrm{Rep}(\mathrm{GL}_{n[f_n\cdot\ell]=n_1[f_1\cdot\ell]+\cdots+n_s[f_s\cdot\ell]}((F^+_{\overline{v}}\otimes\mathbb{R})\times(F^+_v\otimes\mathbb{R})))\\
&\qquad=\mathop{\boxplus}_{n_\ell=n_1}^{n_s}\mathrm{Irr}\,\mathrm{Rep}(\mathrm{GL}_{n_\ell[f_\ell\cdot\ell]}((F^+_{\overline{v}}\otimes\mathbb{R})\times(F^+_v\otimes\mathbb{R})))\ .
\end{aligned}$$

5.17 Proposition

If

$$G^{(2n[2f_n\cdot\ell])}((F^+_{\overline{v}}\otimes\mathbb{R})\times(F^+_v\otimes\mathbb{R})))=\mathop{\boxplus}_{n_\ell=n_1}^{n_s}G^{(2n_\ell[2f_\ell\cdot\ell])}((F^+_{\overline{v}}\otimes\mathbb{R})\times(F^+_v\otimes\mathbb{R})))$$

represents the decomposition of the shifted $2n$-dimensional real bilinear complete semigroup into irreducible components of dimension $2n_\ell$ shifted in $2f_\ell\cdot\ell$ dimensions, then the $2n$-dimensional reducible shifted global Weil representation of the shifted bilinear global Weil group $(W^{ab}_{F^{+,(S_{\mathbb{R}})}_{\overline{v}}}\times W^{ab}_{F^{(+,(S_{\mathbb{R}})}_{v}})$ *is given by:*

$$\begin{aligned}
\mathrm{RedRep}^{(2n[2f_n\cdot\ell])}_{W_{F^+_{R\times L}}}(W^{ab}_{F^{+,(S_{\mathbb{R}})}_{\overline{v}}}\times W^{ab}_{F^{(+,(S_{\mathbb{R}})}_{v}})&=G^{(2n[2f_n\cdot\ell])}((F^+_{\overline{v}_\oplus}\otimes\mathbb{R})\times(F^+_{v_\oplus}\otimes\mathbb{R})))\\
&=\bigoplus_{n_\ell=n_1}^{n_s}\mathrm{Irr}\,\mathrm{Rep}^{(2n_\ell[2f_\ell\cdot\ell])}_{W_{F^+_{R\times L}}}(W^{ab}_{F^{+,(S_{\mathbb{R}})}_{\overline{v}}}\times W^{ab}_{F^{(+,(S_{\mathbb{R}})}_{v}})\ .
\end{aligned}$$

5.18 Proposition

The toroidal compactification of $G^{(2n[2f_n\cdot\ell])}((F^+_{\overline{v}_\oplus}\otimes\mathbb{R}\,)\times(F^+_{v_\oplus}\otimes\mathbb{R}\,)))$ generates by decomposition:

$$G^{(2n[2f_n\cdot\ell])}((F^{+,T}_{\overline{v}_\oplus}\otimes\mathbb{R}\,)\times(F^{+,T}_{v_\oplus}\otimes\mathbb{R}\,)))=\bigoplus_{n_\ell=n_1}^{n_s}G^{(2n_\ell[2f_\ell\cdot\ell])}((F^{+,T}_{\overline{v}_\oplus}\otimes\mathbb{R}\,)\times(F^{+,T}_{v_\oplus}\otimes\mathbb{R}\,)))$$

whose elliptic representation is given by:

$$\mathrm{RedELLIP}(\mathrm{GL}_{n[f_n\cdot\ell]}((F^{+,T}_{\overline{v}}\otimes\mathbb{R}\,)\times(F^{+,T}_{v}\otimes\mathbb{R}\,)))=\bigoplus_{n_\ell=n_1}^{n_s}\mathrm{ELLIP}_{R\times L}(2n_\ell[2f_\ell\cdot\ell],j_\delta,m_{j_\delta})$$

where $\mathrm{ELLIP}_{R\times L}(2n_\ell[2f_\ell\cdot\ell],j_\delta,m_{j_\delta})$ is the product, right by left, of $2n_\ell$-dimensional shifted global elliptic semimodules as introduced in section 5.12.

5.19 Proposition

Let

$$\partial\overline{S}^{P_{n[f_n\cdot\ell]}}_{\mathrm{GL}_{n[f_n\cdot\ell]}}=\boxplus_{n_\ell=n_1}^{n_s}\partial\overline{S}^{P_{n_\ell[f_\ell\cdot\ell]}}_{\mathrm{GL}_{n_\ell[f_\ell\cdot\ell]}}$$

be the decomposition of the boundary $\partial\overline{S}^{P_{n[f_n\cdot\ell]}}_{\mathrm{GL}_{n[f_n\cdot\ell]}}$ of the reducible shifted bisemispace $\overline{S}^{P_{n[f_n\cdot\ell]}}_{\mathrm{GL}_{n[f_n\cdot\ell]}}$ into irreducible components $\partial\overline{S}^{P_{n_\ell[f_\ell\cdot\ell]}}_{\mathrm{GL}_{n_\ell[f_\ell\cdot\ell]}}$. Then, the Eisenstein cohomology of this reducible shifted bisemispace $\partial\overline{S}^{P_{n[f_n\cdot\ell]}}_{\mathrm{GL}_{n[f_n\cdot\ell]}}$ decomposes following:

$$\begin{aligned}H^*(\partial\overline{S}^{P_{n[f_n\cdot\ell]}}_{\mathrm{GL}_{n[f_n\cdot\ell]}},\widehat{M}^{2n}_{T_{R_\oplus}}[2f_n\cdot\ell]\otimes\widehat{M}^{2n}_{T_{L_\oplus}}[2f_n\cdot\ell])&\\&=\bigoplus_{n_\ell=n_1}^{n_s}H^{2n_\ell-2f_\ell\cdot\ell}(\partial\overline{S}^{P_{n[f_n\cdot\ell]}}_{\mathrm{GL}_{n[f_n\cdot\ell]}},\widehat{M}^{2n_\ell}_{T_{v_R}}[2f_\ell\cdot\ell]\otimes\widehat{M}^{2n_\ell}_{T_{v_L}}[2f_\ell\cdot\ell])\\&\simeq\bigoplus_{n_\ell}\mathrm{CY}^{2n_\ell}(\partial\overline{Y}_R,[2f_\ell\cdot\ell])\times\mathrm{CY}^{2n_\ell}(\overline{Y}_L,[2f_\ell\cdot\ell])\end{aligned}$$

where $(\widehat{M}^{2n}_{T_{v_R}}[2f_n\cdot\ell]\otimes\widehat{M}^{2n}_{T_{v_L}}[2f_n\cdot\ell])$ is the bisemisheaf over the partially reducible $\mathrm{GL}_{n[f_n\cdot\ell]}((F^{+,T}_{\overline{v}}\otimes\mathbb{R}\,)\times(F^{+,T}_{v}\otimes\mathbb{R}\,))$-bisemimodule.

5.20 Proposition

On the reducible shifted pseudoramified bilinear complete semigroup

$$G^{(2n[2f_n\cdot\ell])}((F^+_{\overline{v}}\otimes\mathbb{R}\,)\times(F^+_{v}\otimes\mathbb{R}\,))=\boxplus_{n_\ell=n_1}^{n_s}G^{(2n_\ell[2f_\ell\cdot\ell])}((F^+_{\overline{v}}\otimes\mathbb{R}\,)\times(F^+_{v}\otimes\mathbb{R}\,))$$

there exists the ***geometric-shifted global bilinear "reducible" correspondence of Langlands:***

$$\begin{tikzcd}
\mathrm{RedRep}^{(2n[2f_n\cdot\ell])}_{W_{F^+_{R\times L}}}(W^{ab}_{F^{+,(S_{\mathbb{R}})}_{\overline{v}}}\times W^{ab}_{F^{+,(S_{\mathbb{R}})}_{v}}) \arrow[r, "\sim"] \arrow[d, equal] & \mathrm{RedELLIP}(\mathrm{GL}_{n[f_n\cdot\ell]}((F^{+,T}_{\overline{v}}\otimes\mathbb{R})\times(F^{+,T}_{v}\otimes\mathbb{R}))) \arrow[d, equal] \\
G^{(2n[2f_n\cdot\ell])}((F^{+}_{\overline{v}_{\oplus}}\otimes\mathbb{R})\times(F^{+}_{v_{\oplus}})\otimes\mathbb{R}) \arrow[dr, "\sim"] & \bigoplus\limits_{n_\ell}\mathrm{ELLIP}_{R\times L}(2n_\ell[2f_\ell\cdot\ell],j_\delta,m_{j_\delta}) \\
 & G^{(2n[2f_n\cdot\ell])}((F^{+,T}_{\overline{v}}\otimes\mathbb{R})\times(F^{+,T}_{v}\otimes\mathbb{R})) \arrow[u, "\wr"] \arrow[d, equal] \\
 & H^*(\partial\overline{S}^{P_{n[f_n\cdot\ell]}}_{\mathrm{GL}_{n[f_n\cdot\ell]}},\widehat{M}^{2n}_{T_{v_{R_\oplus}}}[2f_n\cdot\ell]\otimes\widehat{M}^{2n}_{T_{v_{L_\oplus}}}[2f_n\cdot\ell])
\end{tikzcd}$$

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